LONGMANS'

SCHOOL MENSURATION

WITH ANSWERS

PEARCE

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BY

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WITH AN ADDITIONAL CHAPTER AND EXERCISES

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PREFACE

Mensuration is so important a subject, and its results are of such practical utility in everyday life, that the ordinary course of arithmetic often is, and always should be, followed by the addition of rules and exercises on the subject.

The following pages, it is hoped, will supply a want frequently felt by those who have had experience in teaching the subject; and this book will be found to differ from those manuals most generally used in the following particulars:—

(*a*) Except in one or two unimportant cases, where a knowledge of the higher mathematics is necessary, a simple proof of every rule is given which can be easily mastered by all students who have a good knowledge of arithmetic and an elementary knowledge of algebra and geometry.

(*b*) The diagrams illustrating the various figures and solids are very numerous, and have been carefully prepared with a view to elucidate the text.

(*c*) In addition to a very large number of examples at the end of each section, several sets of examination papers have been introduced at convenient stages.

(*d*) A set of very easy questions has been inserted at the end. These can be used either for mental work, rapid revision, or as easy exercises for beginners.

It is hoped that the treatment of the subject as herein exemplified will lead to a more intelligible knowledge of the principles on which the rules of mensuration are based, and that, besides supplying exercises for arithmetical accuracy, it will have an educational advantage, and render the subject an effective means for mental discipline.

The number of examples is so numerous that it can hardly be hoped the answers will be free from error. Great care has been taken, however, to ensure accuracy. The author will be thankful to receive any corrections or suggestions relating to the book.

Rose Hill School, Bowdon,
April, 1892.

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LONGMANS' SCHOOL MENSURATION.

BOOK I.

MENSURATION treats of the measurement of lengths, areas, and volumes.

I.—ON THE MEASUREMENT OF LENGTHS.

Lengths or distances are measured by the foot, yard, or mile. Land is measured by the chain, which consists of 100 links, and is 22 yards in length.

Table of Long Measure.

12 in. = 1 ft.
36 in. = 3 ft. = 1 yd.
$16\frac{1}{2}$ ft. = $5\frac{1}{2}$ yds. = 1 po.
660 ft. = 220 yds. = 40 po. = 1 fur.
5280 ft. = 1760 yds. = 320 po. = 8 fur. = 1 mile.
3 miles = 1 league.

Table of Chain Measure.

100 lks. = 22 yds. = 1 ch.
220 yds. = 10 ch. = 1 fur.
1760 yds. = 80 ch. = 8 fur. = 1 mile.

Example.—How many chains will measure a distance of 5 miles 3 fur. 15 po.?

$$5 \text{ miles } 3 \text{ fur. } 15 \text{ po.} = 1735 \times 5\tfrac{1}{2} \text{ yds.} = \frac{1735 \times 5\frac{1}{2}}{22} \text{ chains.}$$

$$= \frac{1735}{4} \text{ ch.} = 433\tfrac{3}{4} \text{ ch. Ans.}$$

(*See* EASY EXERCISES I. A.)

EXERCISE I.

1. Reduce 46 miles 5 fur. 24 po. 3 yds. to yards.
2. Reduce 839067 yds. to miles, etc.
3. In $14\frac{3}{4}$ miles how many feet?
4. How many chains in 4 miles 3 fur.?
5. How many miles, etc., in 765 ch.?
6. What distance in yards is $15\frac{1}{2}$ ch.?
7. How many chains are there in 13 miles 3 fur.?
8. In seven million inches how many miles, etc.?
9. What is the difference in yards between $9\frac{1}{2}$ miles and $54\frac{1}{2}$ fur.?
10. What is the difference in feet between $1\frac{3}{4}$ miles and 135 ch.?
11. In thirty thousand links how many miles?
12. How many chains in 5 miles 1 fur. 20 po.?

II.—ON THE MEASUREMENT OF AREAS.

An area is a surface enclosed by one or more boundaries or lines.

Areas are measured by—

(i.) Square measure, sometimes called land measure.
(ii.) Square chain measure.

(i.)—Table of Square Measure (Land).

12 in. × 12 in.
= 144 sq. in. = 1 sq. ft.
3 ft. × 3 ft. = 9 sq. ft. = 1 sq. yd.
$5\frac{1}{2}$ yds. × $5\frac{1}{2}$ yds. = $30\frac{1}{4}$ sq. yds. = 1 sq. po.
1210 sq. yds. = 40 sq. po. = 1 ro.
4840 sq. yds. = 160 sq. po. = 4 ro. = 1 ac.

(ii.)—Table of Square Chain Measure.

100 lks. × 100 lks.
= 10000 sq. lks. = 1 sq. ch. = 484 sq. yds.
100000 sq. lks. = 10 sq. ch. = 4840 sq. yds. = 1 ac.
640 ac. = 1 sq. mile.

Example I.—How many acres in 3507025 sq. lks.?

$$3507025 \text{ sq. lks.} = 35{\cdot}07025 \text{ ac.}$$
$$\begin{array}{r} 4 \\ \hline {\cdot}28100 \text{ ro.} \\ 40 \\ \hline 11{\cdot}240 \text{ po.} \end{array}$$

Ans. 35 ac. 0 ro. $11\frac{6}{25}$ po.

Example II.—What is the measure of a field of 3 ac. when the square whose side is 11 yds. is the unit?

$$3 \text{ ac.} = 4840 \text{ sq. yds.} \times 3$$
$$\text{The unit of area} = 11 \times 11 = 121 \text{ sq. yds.}$$
$$\therefore \text{No. of units} = \frac{4840 \times 3}{121} = 120 \quad \text{Ans.}$$

(*See* Easy Exercises I. B.)

EXERCISE II.

1. Reduce 17 ac. 3 ro. 34 po. to square yards.
2. In 905371 sq. yds. how many acres?
3. Reduce $33\frac{3}{4}$ ac. to square yards.
4. Reduce 768595 sq. lks. to acres, etc.
5. How many square links in $6\frac{3}{4}$ ac.?
6. How many acres, etc., in 549 sq. ch.?
7. Reduce 15 ac. 3 ro. to square links.
8. Reduce to acres, etc., 5390785 sq. lks.
9. How many square yards in $25\frac{1}{2}$ ac.?
10. How many square yards in 58129 sq. in.?
11. In 29 ac. 3 ro. 15 po. how many square yards?
12. How many acres, roods, etc., in 4050500 sq. lks.
13. How many acres, etc., in $709\frac{1}{2}$ sq. ch.?
14. If the unit of measure is 1100 sq. yds., what is the area of a field in acres, etc., whose measure is 22?
15. What is the measure of an acre when a square whose side is 22 yds. is the unit?
16. What is the unit of measure of a field when a field of 10 ac. measures 242?

III.—ON THE MULTIPLICATION OF MEASUREMENTS.

The product arising from the multiplication of measurements may be found in various ways.

(i.) By FRACTIONS.

Example.—Required the product of 5 ft. 8 in. by 10 ft. 10 in.

$$5 \text{ ft. } 8 \text{ in.} \times 10 \text{ ft. } 10 \text{ in.} = 5\tfrac{2}{3} \text{ ft.} \times 10\tfrac{5}{6} \text{ ft.} = \tfrac{17}{3} \text{ ft.} \times \tfrac{65}{6} = \tfrac{1105}{18} \text{ sq. ft.}$$
$$= 61\tfrac{7}{18} \text{ sq. ft.} = 61 \text{ sq. ft. } 56 \text{ sq. in.} \quad \text{Ans.}$$

(ii.) By REDUCTION.

Example.—Required the product of 4 ft. 7 in. by 8 ft. 11 in.

$$4 \text{ ft. } 7 \text{ in.} \times 8 \text{ ft. } 11 \text{ in.} = 55 \text{ in.} \times 107 \text{ in.} = 5885 \text{ sq. in.}$$
$$= 40 \text{ sq. ft. } 125 \text{ sq. in.} \quad \text{Ans.}$$

(iii.) By DUODECIMALS, sometimes called CROSS MULTIPLICATION.

Table of Duodecimals.

12 sq. in. = 1 twelfth of a sq. ft. or 1 superficial prime.
144 sq. in. = 12 twelfths of a sq. ft. or 12 superficial primes = 1 sq. ft.

A *square inch* is a square surface measuring an inch long and an inch broad.

A *superficial prime*, or one-twelfth of a square foot, is a rectangular surface measuring one foot in length by one inch in breadth.

A *square foot* is a square surface measuring one foot in length and one foot in breadth.

Therefore the product of
(i.) Inches by inches = square inches.
(ii.) Inches by feet = superficial primes.
(iii.) Feet by feet = square feet.

Example.—Required the product of 7 ft. 8 in. by 8 ft. 7 in.

ft.	in.
7	8
8	7

61 sq. ft. 4 primes = 7 ft. 8 in. × 8 ft.
4 sq. ft. 5 primes 8 sq. in. = 7 ft. 8 in. × 7 in.
65 sq. ft. 9 primes 8 sq. in. = 7 ft. 8 in. × 8 ft. 7 in.

Ans. 65 sq. ft. 116 sq. in.

(*See* EASY EXERCISES I. C.)

EXERCISE III. (A.)

1. Multiply 5 ft. 11 in. by 4 ft. 7 in.
2. „ 5 ft. 7 in. by 4 ft. 10 in.
3. „ 4 ft. 5 in. by 3 ft. 9 in.
4. „ 18 ft. 9 in. by 14 ft. 7 in.
5. „ 23 ft. 8 in. by 16 ft. 9 in.
6. „ 2 ft. 9 in. by 10 ft. 4 in.
7. „ 17 ft. 3 in. by 13 ft. 10 in.
8. „ 9 ft. 3 in. by 3 ft. 5 in.
9. „ 15 ft. 9 in. by 12 ft. 4 in.
10. „ 10 yds. 2 ft. 5 in. by 6 yds. 1 ft. 8 in.
11. „ 15 yds. 1 ft. 10 in. by 9 yds. 9 in.
12. „ 6 yds. 2 ft. 8 in. by 5 yds. 6 in.

IV.—ON THE DIVISION OF MEASUREMENTS.

The quotient arising from the division of the measurement of a surface or area by a length must give a measurement in length.

Thus (i.) square feet divided by feet give feet.
(ii.) square inches divided by inches give inches.
(iii.) square chains divided by chains give chains.

The results, as in multiplication, are obtained in several ways.

(i.) By Fractions.

Example.—Required the quotient of 36 sq. ft. 96 sq. in. by 6 ft. 8 in.

36 sq. ft. 96 sq. in. ÷ 6 ft. 8 in. = $36\frac{2}{3}$ sq. ft. ÷ $6\frac{2}{3}$ ft.
= $\frac{110}{3}$ sq. ft. ÷ $\frac{20}{3}$ ft. = $\frac{11}{2}$ ft. = $5\frac{1}{2}$ ft. Ans.

(ii.) By Reduction.

Example.—Required the quotient of 106 sq. ft. 36 sq. in. by 8 ft. 4 in.

106 sq. ft. 36 sq. in. ÷ 8 ft. 4 in. = 15300 sq. in. ÷ 100 in.
= 153 in. = 12 ft. 9 in. Ans.

EXERCISE III. (B.)

1. Divide 29 sq. ft. 56 sq. in. by 7 ft. 8 in.
2. „ 2 sq. yds. 6 sq. ft. by 5 ft. 4 in.
3. „ 35 sq. yds. 5 sq. ft. 48 sq. in. by 20 ft. 8 in.

4. Divide 21 ac. 1 ro. 30 po. by 385 yds.
5. „ $3\frac{1}{2}$ ac. by 242 yds.
6. „ 1 sq. mile by 5 miles.
7. „ $5\frac{1}{4}$ ac. by 32 ch.
8. „ 3 ac. 34 po. by 5 ch. 14 lks.
9. „ 180 sq. yds. 4 sq. ft. by 9 yds. 2 ft.
10. „ 127 sq. yds. 4 sq. ft. by 12 yds. 1 ft.
11. „ 59 sq. ft. by $2\frac{1}{2}$ sq. ft.
12. „ 90 sq. in. by $4\frac{1}{2}$ sq. in.

V.—ON THE RIGHT-ANGLED TRIANGLE.

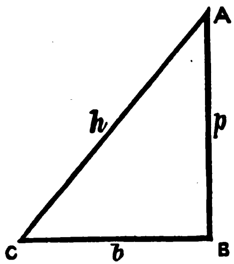
A *triangle* is a figure bounded by three straight lines.

A *right-angled triangle* is one which has one of its angles a right angle.

The side opposite the right angle is called the *hypothenuse.*

The other two sides are called the *base* and *perpendicular* respectively.

(*a*) To find the hypothenuse, the base and perpendicular being given.



Let ABC be a right-angled triangle, having ABC the right angle.

Let AC the hypothenuse = h.

Let AB the perpendicular = p.

Let BC the base = b.

Then $h^2 = p^2 + b^2$ (Euc. I. 47.)

$\therefore h = \sqrt{p^2 + b^2}$

Rule.—**Take the square root of the sum of the squares of the base and perpendicular.**

(*b*) To find the base when the hypothenuse and perpendicular are given.

Now $p^2 + b^2 = h^2$

$\therefore b^2 = h^2 - p^2 = (h + p)(h - p)$

$\therefore b = \sqrt{h^2 - p^2} = \sqrt{(h + p)(h - p)}$

Rule.—**Take the square root of the difference of the squares of the hypothenuse and perpendicular;** *or,* **Take the square root of the product of the sum and difference of the hypothenuse and perpendicular.**

(*c*) **To find the perpendicular when the hypothenuse and base are given.**

$$\text{Now } p^2 + b^2 = h^2$$
$$\therefore p^2 = h^2 - b^2 = (h + b)(h - b)$$
$$\therefore p = \sqrt{h^2 - b^2} = \sqrt{(h + b)(h - b)}$$

RULE.—**Take the square root of the difference of the squares of the hypothenuse and base ; *or*, Take the square root of the product of the sum and difference of the hypothenuse and base.**

Example.—The length of the hypothenuse is 27 ft., and of the base of a right-angled triangle is 15 ft. : what is the length of the perpendicular?

$$p = \sqrt{h^2 - b^2} = \sqrt{27^2 - 15^2} = \sqrt{(27 + 15)(27 - 15)}$$
$$= \sqrt{42 \times 12} = \sqrt{7 \times 6 \times 6 \times 2} = 6\sqrt{14} = 22{\cdot}44 \text{ ft.} \quad \text{Ans.}$$

EXERCISE IV.

Find the hypothenuse of the following right-angled triangles, whose measurements are—

1. Base 3 ft. 4 in., perpendicular 2 ft. 8 in.

2. Base 7584 ft., perpendicular 3937 ft.

3. Base 2 ch. 20 lks., perpendicular 1 ch. 63 lks.

4. Base $15\frac{1}{2}$ yds., perpendicular $16\frac{1}{2}$ yds.

Find the remaining side of the following right-angled triangles, whose measurements are—

5. Hypothenuse 6 ft. 9 in., base 3 ft. 8 in.

6. Hypothenuse $8\frac{1}{2}$ fur., perpendicular $6\frac{3}{4}$ fur.

7. Hypothenuse 725 ft., perpendicular 644 ft.

8. Hypothenuse 3 ch. 13 lks., base 1 ch.

9. A ladder 36 ft. long stands erect close to the wall of a building. How many inches will the top fall if the foot be pulled out 13 ft. from the wall?

10. Barcelona is 188 miles N.E. from Valencia, and 570 miles N.W. from Tunis : find the distance of Tunis from Valencia.

11. The sides of a garden in the form of a right-angled triangle are 133 ft. and 156 ft. respectively : find the length of the hypothenuse.

12. One end of a rope 52 ft. long is tied to the top of a pole 48 ft. high, and the other end is fastened to a peg in the ground. If the pole be vertical and the rope tight, find how far the peg is from the foot of the pole.

13. A wall 72 ft. high is built at one edge of a moat 54 ft. wide : how long must scaling-ladders be to reach from the other edge of the moat to the top of the wall?

14. The houses in a street are 40 ft. high, and the street is 30 ft. wide:

find the length of the ladder which will reach from the top of one of the houses to the opposite side of the street.

15. Find the length of the perpendicular drawn from the vertex of an isosceles triangle whose equal sides are 10 ft. and whose base is 2 ft.

16. A flagstaff 30 ft. high was broken over, and the top struck the ground 18 ft. from the bottom of the staff: what was the length of the part which fell?

VI.—A. ON SURFACES.

Definitions.

A *parallelogram* is a figure bounded by four straight lines whose opposite sides are parallel.

A *rectangular parallelogram* is one all of whose angles are right angles. It is also called a *rectangle.*

A *square* is a rectangular parallelogram which has all its sides equal.

An *oblique parallelogram* is one having none of its angles right angles.

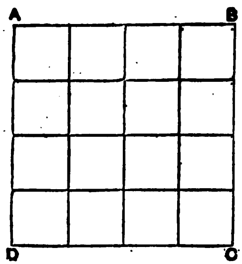
A *rhombus* is an oblique parallelogram having all its sides equal.

A *rhomboid* is an oblique parallelogram having its opposite sides equal.

VI.—B. ON RECTANGULAR PARALLELOGRAMS.

(i.) The Square.

(*a*) To find the area, side being given.



Let ABCD be a square whose side AB measures 4 units. It is evident that the surface contains 16 square units.

Let A = area, and s = side.

Then $A = s^2$.

Rule.—**Take the square of the given side.**

(*b*) To find the perimeter, the side being given.

The *perimeter* of a figure is the sum of its boundaries.

∴ Perimeter of a square = 4 times the length of the side.

Rule.—**Multiply the length of the side by 4.**

(c) To find the side, area being given.

$$\text{Now } s^2 = \text{A}$$
$$\therefore s = \sqrt{\text{A}}.$$

RULE.—**Take the square root of the area.**

(d) To find the diagonal, side being given.

The diagonal of a square is the hypothenuse of a right-angled triangle whose base and perpendicular are the sides of the square.

Let d = diagonal, and s = side.

$$\text{Then } d^2 = s^2 + s^2 = 2s^2$$
$$\therefore d = \sqrt{2s^2} = s\sqrt{2}$$

RULE.—**Multiply the length of the side by the square root of 2.**

(e) To find the diagonal, area given.

$$\text{Now } d^2 = 2s^2 \text{ from above}$$
$$\text{And } s^2 = \text{A}$$
$$\therefore d^2 = 2\text{A}$$
$$\therefore d = \sqrt{2\text{A}}.$$

RULE.—**Take the square root of twice the area.**

(f) To find the area, diagonal given.

$$\text{Now } 2s^2 = d^2$$
$$\therefore s^2 = \frac{d^2}{2}$$
$$\text{But } s^2 = \text{A}$$
$$\therefore \text{A} = \frac{d^2}{2}$$

RULE.—**Take one-half of the square of the diagonal.**

Example.—The rent of a square field at £2 14*s*. 6*d*. per acre amounts to £27 5*s*. Find the cost of putting a paling round the field at 9*d*. a yard.

1 acre costs £2 14*s*. 6*d*.

$\therefore$ No. of acres = £27 5*s*. ÷ £2 14*s*. 6*d*.

$\therefore$ Area in square yards $= 27\frac{1}{4} \div 2\frac{29}{40} \times 4840 = \frac{109}{4} \times \frac{40}{109} \times 4840 = 48400$

$\therefore$ Length of one side in yards $= \sqrt{48400} = 220$ yds.

$$\therefore \text{Cost of paling} = \frac{220 \times 4 \times 3}{4} = 660s. = £33 \quad \text{Ans.}$$

(*See* EASY EXERCISES II. B AND C.)

EXERCISE V.

Find the area in square yards of squares whose sides are—

1. 6½ yds. **2.** 13 yds. **3.** 30¼ yds.
4. 83 ft. **5.** 10 yds. 2 ft. **6.** 15 ft. 8 in.

Find the area in square yards, etc., of squares whose sides are—

7. 12 yds. 2 ft. **8.** 14 yds. 1 ft. **9.** 3 yds. 1 ft.

Find the area in acres, etc., of squares whose sides are—

10. 500 ft. **11.** 110 yds. **12.** 121 yds.

Find the area in acres, etc., of squares, whose sides are—

13. 12 ch. 25 lks. **14.** 18 ch. 36 lks.
15. 3250 lks. **16.** 26 ch. 35 lks.

Find the sides of squares having the following areas :—

17. 8649 sq. yds. **18.** 6¼ ac.
19. 10 ac. 3 ro. 20 po. **20.** 64·064016 sq. ft.

Find the areas of the squares w e diagonals are—

21. 275 yds. **22.** 1760 yds. **23.** 9 ft. 9 in. **24.** 1000 lks.

Find the diagonals of the squares whose areas are—

25. 7 sq. in. **26.** 1521 sq. yds. **27.** 5 ac. **28.** 1½ ac.

29. Find the length of the side of a square enclosure the paving of which cost £27 1*s*. 6*d*. at 8*d*. a square yard.

30. Find the side of a square field containing 2 ac. 121 sq. yds.

31. Find the cost of turfing a square tennis-court whose side is 42 ft., at 6½*d*. a square yard.

32. Find the area of a square whose side is 871 lks.

33. The sides of three squares being 5, 6, and 7 ft. respectively, find the side of a square which is equal in area to the sum of the three.

34. What is the area of a square field whose side is 15 ch. 40 lks. ?

35. Find the side of a square field containing 10 ac.

36. What is the length of the side of a square garden that costs £33 16*s*. 10½*d*., trenching at 2½*d*. per square yard ?

37. The diagonal of a square courtyard is 30 yds. : find the cost of gravelling it at 10½*d*. for 9 yds.

38. The rent of a square field at £2 14*s*. 6*d*. an acre amounts to £27 5*s*. : find the cost of putting a paling round it at 7½*d*. a yard.

39. The diagonal of a square field is 875 lks. : find the side of another square field which contains three times the area.

40. The diagonal of the floor of a square room measures 32 ft., and the height of the room is 14 ft. How many yards of paper 9·45 in. wide will be required to cover the walls, allowing 15 sq. yds. for door, windows, etc. ?

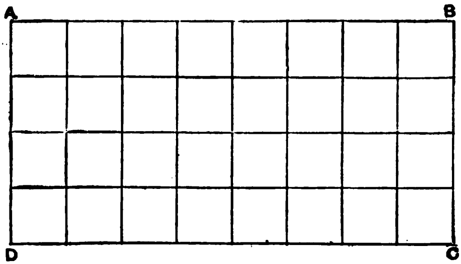
41. Of two squares, one measures 44 ft. more round than the other, and 187 sq. ft. more in area. What are their respective sizes?

(ii.) The Rectangle (or Oblong).

(*a*) To find the area of a rectangle.

Let ABCD be a rectangle.

If AB = 8 units, and BC = 4 units, it is evident the area of the rectangle equals 8×4 sq. units = 32 sq. units.



Let A = number of square units in area, l = number of units in length, and b = number of units in breadth.

Then $A = l \times b$

Rule.—**Multiply the length by the breadth.**

(*b*) To find the perimeter, length and breadth given.

Let P = perimeter.

Then $P = 2(l + b)$

Rule.—**Multiply the sum of the length and breadth by two.**

(*c*) To find the other side, area and one side being given.

$$A = b \times l$$

$$\therefore \text{(i.) } b = \frac{A}{l}$$

$$\text{(ii.) } l = \frac{A}{b}$$

Rule.—**Divide the area by the given side.**

(*d*) To find the diagonal, length and breadth being given.

Let d = diagonal

Then $d^2 = l^2 + b^2$ (Euc. I. 47, and Sect. V. (*a*).)

$\therefore\ d = \sqrt{l^2 + b^2}$

RULE.—**Take the square root of the sum of the squares of the length and breadth.**

Example.—Find the cost of surrounding a bowling-green 80 ft. by 46 ft. 2 in. with a paved walk a yard and a half wide, at 2*s.* 8*d.* per square foot.

$$\text{Area of bowling-green and walk} = (80 + 9) \times (46\tfrac{1}{6} + 9) = 89 \times 55\tfrac{1}{6} \text{ sq.}$$
$$\text{Area of bowling-green} = 80 \times 46\tfrac{1}{6} \text{ sq. ft.}$$
$$\therefore \text{Area of the walk} = 89 \times 55\tfrac{1}{6} - 80 \times 46\tfrac{1}{6} = \tfrac{7299}{6} \text{ sq. ft.}$$
$$\therefore \text{Cost of the walk} = \tfrac{7299}{6} \times 2\tfrac{2}{3}s. = \tfrac{7299}{6} \times \tfrac{8}{3} = 3244s.$$
$$= £162\ 4s. \quad \text{Ans.}$$

(iii.) ON THE AREAS OF FLOORS AND THE WALLS OF ROOMS.

(*a*) *Floors of rooms* are generally squares or rectangles, and their areas will be found by multiplying the length by the breadth.

(*b*) *Walls of rooms* are also generally squares or rectangles, and their areas will be found by multiplying the total length of the walls by the height of the room.

(*c*) *The length of carpet* required to cover a floor equals the area of the room divided by the width of the carpet.

Example.—A room is 15 ft. 9 in. long, 12 ft. 8 in. broad, and 11 ft. high: find the area of the floor, and of the walls, and the number of yards of carpet 2 ft. 3 in. wide required to cover it.

$$\text{(i.) Area of floor} = 15 \text{ ft. } 9 \text{ in.} \times 12 \text{ ft. } 8 \text{ in.} = 15\tfrac{3}{4} \times 12\tfrac{2}{3} \text{ sq. ft.}$$
$$= \frac{63}{4} \times \frac{38}{3} = \frac{399}{2} = 199\tfrac{1}{2} \text{ sq. ft.} \quad \text{Ans.}$$

$$\text{(ii.) Area of walls} = 2\,(15 \text{ ft. } 9 \text{ in.} + 12 \text{ ft. } 8 \text{ in.}) \times 11 \text{ ft.}$$
$$= 28 \text{ ft. } 5 \text{ in.} \times 2 \times 11 \text{ ft.}$$
$$= 28\tfrac{5}{12} \times 2 \times 11 = \frac{341 \times 2 \times 11}{12} = \frac{3751}{6} \text{ sq. ft}$$
$$= 625\tfrac{1}{6} \text{ sq. ft.} \quad \text{Ans.}$$

$$\text{(iii.) Length of carpet} = 15\tfrac{3}{4} \times 12\tfrac{2}{3} \div 2\tfrac{1}{4} = \frac{63}{4} \times \frac{38}{3} \times \frac{4}{9} = \frac{266}{3} \text{ ft.}$$
$$= \frac{266}{9} \text{ yds.} = 29\tfrac{5}{9} \text{ yds.} \quad \text{Ans.}$$

(*See* EASY EXERCISES II. A AND D.)

EXERCISE VI.

Find the areas in square yards of the following rectangles, whose sides are—

1. 17 ft. by 3 yds. 1 ft.
2. 10 yds. 1 ft. by 12 yds. 1 ft.
3. 7 yds. 2 ft. by 5 yds. 1 ft.
4. 2 yds. 2 ft. by 1 yd. $1\frac{1}{2}$ ft.
5. 6 yds. $1\frac{1}{2}$ ft. by 5 yds. 2 ft.
6. 8 yds. 1 ft. by 7 yds. 1 ft.

Find the areas in acres, etc., of the following rectangular fields, whose sides are—

7. 7 ch. 4 lks. by 8 ch. 12 lks.
8. 10 ch. 80 lks. by 12 ch. 40 lks.
9. 11 ch. 25 lks. by 8 ch. 30 lks.

Find the breadth of the following rectangles, whose measurements are—

10. Area 1 ac., and length 110 yds.
11. Area 1000 ac., and length $2\frac{1}{2}$ miles.
12. Area $5\frac{1}{4}$ ac. and length 32 ch.

Find the perimeters of the following rooms, whose measurements are—

13. Length 12 ft. 8 in., and breadth 10 ft. 9 in.
14. „ 14 ft. 3 in., „ 12 ft. 6 in.
15. „ 14 ft. 7 in., „ 13 ft. 5 in.

Find the areas in square yards of the following rooms, whose measurements are—

16. Length 12 ft. 8 in., and breadth 10 ft. 9 in.
17. „ 14 ft. 3 in., „ 12 ft. 6 in.
18. „ 14 ft. 7 in., „ 13 ft. 5 in.

What is the area in square yards, etc., of the walls of the following rooms, whose measurements are—

19. Length 12 ft. 8 in., breadth 10 ft. 9 in., and height 9 ft. 6 in.
20. „ 14 ft. 3 in., „ 12 ft. 6 in., „ 10 ft. 6 in.
21. „ 14 ft. 7 in., „ 13 ft. 5 in., „ 11 ft. 6 in.

22. A room is 26 ft. long, 17 ft. wide, and 15 ft. high: how many yards of paper 24 in. wide will cover the walls, allowing 10 sq. yds. for door, windows, and fireplace?

23. A rectangular field of 4 ac. 1 ro. $28\frac{3}{4}$ po. is $781\frac{1}{4}$ lks. in length. What is its breadth?

24. The breadth of a room is half as much again as its height; its length is twice its height, and it costs 5 guineas to paint its walls at $1\frac{1}{4}d.$ per square foot. What are the dimensions of the room?

25. A rectangular grass plot measures 320 yds. by 160 yds.; all round it is a gravel path 6 ft. broad. The price for making the grass plot is

6*d.* per square yard. What must be the price of the gravel path per square yard that the path may cost £1183 4*s.* less than the grass plot?

26. A rectangular court is 120 ft. long and 90 ft. broad, and a path of uniform width of 10 ft. runs round it. Find the cost of covering the path with flagstones at 4*s.* 6*d.* a square yard, and the remainder of the court with turf at 6*s.* 6*d.* per 100 sq. ft.

27. A room is 20 yds. 1 ft. 6 in. long, 15 yds. 1 ft. 6 in. wide, and 17 ft. 6 in. high: how much will it cost to paper it with paper $\frac{3}{4}$ yd. wide at $2\frac{1}{4}$*d.* per yard?

28. The sides of a rectangle are 16 ft. and 10 ft. Find to four places of decimals the length of the diagonal of a square whose area equals that of the rectangle.

29. Find what length of carpet $\frac{3}{4}$ yd. wide will cover a floor 36 ft. 9 in. by 24 ft. 6 in.

30. Find the cost of covering a floor 24 ft. 10 in. by 16 ft. 6 in. with carpet $\frac{3}{4}$ yd. wide at 5*s.* 6*d.* a yard.

31. An oblong courtyard measuring 35 ft. by 24 ft. is paved with oblong slabs each $2\frac{1}{2}$ ft. by $1\frac{3}{4}$ ft.: how many slabs were required?

32. The cost of a carpet for a room $28\frac{1}{2}$ ft. long is £17 8*s.* 4*d.* at 5*s.* 6*d.* a square yard: find the breadth of the room.

33. A room is 14 ft. 9 in. long, 12 ft. 6 in. wide, and 10 ft. 6 in. high: find (i.) area of the walls; (ii.) area of the floor; (iii.) cost of covering the floor with carpet 2 ft. 3 in. wide at 4*s.* 6*d.* a yard.

34. Find what length of drugget $3\frac{3}{4}$ ft. wide will cover a floor 27 ft. 5 in. by 19 ft. 5 in., and what the cost will be at 4*s.* 6*d.* a yard.

35. What is the area of the walls and ceiling of a room which is 14 ft. 7 in. long, 12 ft. 8 in. wide, and 12 ft. 3. in. high?

36. A room is 15 ft. 6 in. long, 12 ft. 3 in. broad, and 11 ft. high: find—

(i.) Area of the room in square yards.
(ii.) Area of the walls in square yards.
(iii.) Number of yards of carpet 2 ft. 3 in. wide required to cover the floor.
(iv.) Cost of painting the walls and ceiling at 8*d.* a square yard.
(v.) Cost of covering the floor with carpet 4 ft. 6 in. wide at 5*s.* 6*d.* a yard.

37. A room is 14 ft. 8 in. long, 12 ft. 6 in. broad, and 11 ft. 6 in. high: find—

(i.) Area of the walls in square yards.
(ii.) Area of the floor in square yards.
(iii.) Number of yards of carpet 3 ft. 6 in. wide required to cover the floor.
(iv.) Cost of papering the walls at $4\frac{1}{2}$*d.* a square yard.

(v.) Cost of painting the ceiling at 2*s.* 9*d.* a square yard.

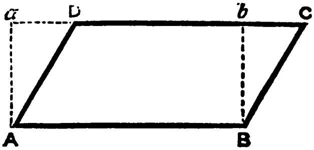
(vi.) Cost of covering the floor with floor-cloth 5 ft. 6 in. wide at 2*s.* 9*d.* a yard.

VII.—ON OBLIQUE PARALLELOGRAMS.

(*a*) To find the area of an oblique parallelogram, the length and perpendicular height being given.

Let ABCD be an oblique parallelogram.

Now, parallelograms on the same base and between the same parallels are equal (Euc. I. 35).



Therefore the area of the oblique parallelogram ABCD equals the area of the rectangle AB*ba*.

Let A = area, l = length, and h = perpendicular height.

Then $A = l \times h$

RULE.—**Multiply the length of the base by the perpendicular height.**

(*b*) To find the perpendicular height, the area and length of base being given.

$$\text{Now } l \times h = A$$

$$\therefore h = \frac{A}{b}$$

RULE.—**Divide the area by the length of the base.**

(*c*) To find the length of base, the area and perpendicular height being given.

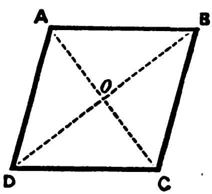
$$\text{Now } l \times h = A$$

$$\therefore l = \frac{A}{h}$$

RULE.—**Divide the area by the perpendicular height.**

(*d*) To find the area of a rhombus, the two diagonals being given.

Let ABCD be a rhombus having the diagonals AC and BD.



The diagonals AC and BC bisect each other in *o* at right angles (Euc. I.).

$$\therefore \text{Area} = \frac{DB(Ao + oC)}{2} = \frac{DB \times AC}{2}$$

(See Sect. VIII. (*a*).)

RULE.—**Take one-half the product of the two diagonals.**

EXERCISE VII.

1. The diagonals of a rhombus are 88 yds. and 110 yds.: find the area.

2. Find the area of a parallelogram whose length of base is 78 ft., and perpendicular height 42 ft.

3. The area of a rhombus is 500 sq. ft., the length of one side is 50 ft.: what is the length of the perpendicular on that side from the opposite side?

4. The diagonals of a rhombus are 25 in. and 15 in.: find the area.

5. Find the area of a rhomboid whose base is 14 yds., and perpendicular height 5 yds.

6. Find the area of a parallelogram whose base is 242 yds., and perpendicular height 70 yds.

7. The area of a parallelogram is 177 sq. yds. 5 sq. ft., and the perpendicular height is 11 yds. 1 ft.: find the base.

8. Each side of a rhombus is 24 ft., and one of the diagonals also is 24 ft.: find the area.

9. Find the area of a rhombus whose base is 12 ft. 10 in., and perpendicular height 9 ft. 8 in.

10. What is the area of a parallelogram whose side is 2185 lks., and perpendicular breadth 1426 lks.?

11. A grass plot in the form of a parallelogram is 50 ft. long and 40 ft. in perpendicular breadth: find the cost of turfing it at 4*d*. per square yard.

12. The sides AB and BC of a parallelogram are 28 and 18: if the perpendicular from A on BC is 10, what must be the perpendicular from C on AB?

13. Find the area of a rhombus whose side is 2 ft. 4 in., and perpendicular breadth 9·32 in.

14. Each side of a rhombus is 65 ft., and one of its diagonals is 104 ft.: find the area.

15. The area of a garden in the form of a rhombus is 99 sq. yds., and its perimeter is 108 ft.: find its perpendicular breadth.

16. The sides of a lawn, which is in the form of a rhombus, are each 50 ft. If the cost of making at 8*d*. per square yard amounts to £7 8*s*. 2*d*., what is the perpendicular breadth?

17. What length of matting $\frac{3}{4}$ yd. wide will cover a room 39 ft. 6 in. long, and whose perpendicular breadth is 25 ft. 6 in.?

18. How much paper 21 in. wide will be required for a room 24 ft. long, 18 ft. wide, and 12 ft. high, allowing for a doorway 8 ft. by $4\frac{3}{8}$ ft. and 3 windows each 6 ft. by $3\frac{1}{2}$ ft.?

VIII.—ON TRIANGLES.

A *triangle* is a figure bounded by three straight lines.

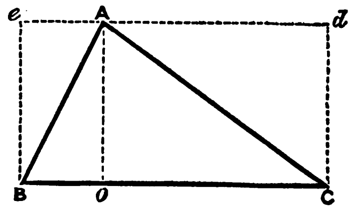
An *equilateral* triangle is one which has its three sides equal.

An *isosceles* triangle is one which has two sides equal.

Any side of a triangle may be regarded as its *base.*

The *perpendicular height* is the perpendicular on the base from the opposite angle.

(*a*) To find the area of a triangle, the length of the base and the perpendicular height being given.



Let ABC be a triangle, and BC*de* a rectangular parallelogram on the same base and between the same parallels BC and *ed.*

Now, a triangle and parallelogram being on the same base and between the same parallels, the triangle is one-half the parallelogram (Euc. I. 41).

Let BC be the base $= b$; and A*o* be the perpendicular height $= h$; and A = area.

$$\text{Then } A = \frac{b \times h}{2}$$

RULE.—**Take one-half of the product of the base by the perpendicular height.**

(*b*) To find the perpendicular height, the area and length of the base being given.

$$\text{Now } \frac{b \times h}{2} = A$$

$$\therefore h = \frac{A \times 2}{b} = \frac{2A}{b}$$

RULE.—**Divide twice the area by the length of the base.**

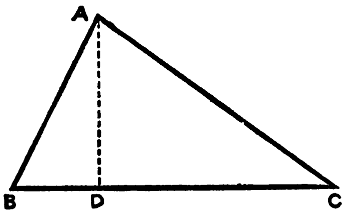
(*c*) To find the length of the base, the area and the perpendicular height being given.

$$\text{Now } \frac{b \times h}{2} = A$$

$$\therefore b = \frac{A \times 2}{h} = \frac{2A}{h}$$

RULE.—**Divide twice the area by the perpendicular height.**

(d) To find the area, the lengths of the three sides being given.



Let ABC be a triangle having its three sides AB = c, BC = a, and AC = b.

Let the perpendicular AD be drawn on BC.

Let BD = x.

Then DC will equal $a - x$.

Now $AD^2 = c^2 - x^2$, and $AD^2 = b^2 - (a - x)^2$ (Euc. I. 47.)

$$\therefore c^2 - x^2 = b^2 - (a - x)^2 = b^2 - a^2 + 2ax - x^2$$

$$\therefore 2ax = a^2 + c^2 - b^2$$

$$\therefore x = \frac{a^2 + c^2 - b^2}{2a} = \text{BD}$$

$$\text{But } AD^2 = c^2 - x^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2$$

$$= \left(c + \frac{a^2 + c^2 - b^2}{2a}\right)\left(c - \frac{a^2 + c^2 - b^2}{2a}\right)$$

$$= \frac{2ac + a^2 + c^2 - b^2}{2a} \times \frac{2ac - a^2 - c^2 + b^2}{2a}$$

$$= \frac{a^2 + 2ac + c^2 - b^2}{2a} \times \frac{b^2 - a^2 + 2ac - c^2}{2a}$$

$$= \frac{(a + c)^2 - b^2}{2a} \times \frac{b^2 - (a - c)^2}{2a}$$

$$= \frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4a^2}$$

$$\therefore AD = \frac{1}{2a}\sqrt{(a + b + c)(a + c - b)(b + a - c)(b + c - a)}$$

$$\text{But area of triangle ABC} = \frac{BC \times AD}{2}$$

$\therefore$ Area of triangle

$$= \frac{a}{2} \times \frac{1}{2a}\sqrt{(a + b + c)(a + c - b)(b + a - c)(b + c - a)}$$

$$= \frac{1}{4}\sqrt{(a + b + c)(a + c - b)(b + a - c)(b + c - a)}$$

Now let $a + b + c = 2s$.

Then $a + c - b$ will equal $2(s - b)$, for $a + b + c - 2b = 2s - 2b$

$$\therefore a + c - b = 2(s - b)$$

And $a + b - c$ will equal $2(s - c)$

And $b + c - a$ will equal $2(s - a)$

∴ Area of triangle

$$= \frac{1}{4}\sqrt{2s \times 2(s-b) \times 2(s-c) \times 2(s-a)}$$

$$= \frac{4}{4}\sqrt{s(s-b)(s-c)(s-a)} = \sqrt{s(s-a)(s-b)(s-c)}$$

Rule.—**From half the sum of the three sides subtract each side separately; multiply the half-sum and the three remainders together, and extract the square root of the product.**

Thus let the lengths of the three sides be a, b, and c respectively Then—

I.—**Find half the sum of the three sides.**

$$\frac{a+b+c}{2} = s$$

II.—**Subtract from this half-sum each side separately.**

$$s - a = r_1 \qquad s - b = r_2 \qquad s - c = r_3$$

III.—**Multiply the half-sum and the three remainders together.**

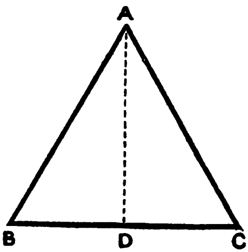
$$s \times r_1 \times r_2 \times r_3$$

IV.—**Take the square root of the product.**

$$\sqrt{s.r_1.r_2.r_3} = \text{Area.}$$

(e) To find the area of an equilateral triangle, the length of the side being given.

Let ABC be an equilateral triangle.
Let BC = s, and AD = h.



Then area of ABC $= \frac{\text{BC} \times \text{AD}}{2} = \frac{s \times h}{2}$

But $h^2 = s^2 - \frac{s^2}{4} = \frac{4s^2 - s^2}{4} = \frac{3s^2}{4}$

$\therefore h = \sqrt{\frac{3s^2}{4}} = \frac{s}{2}\sqrt{3}$

$\therefore$ Area of ABC $= \frac{s \times \frac{s}{2}\sqrt{3}}{2} = \frac{s^2}{4}\sqrt{3}$

Rule.—**Square the length of the given side, and multiply by one-fourth of the square root of 3.**

Note.—The square root of 3 is 1·732, and one-fourth of the square root of 3 is ·433.

(*See* Easy Exercises III., A, B, and C.)

EXERCISE VIII.

Find the area of the following triangles whose measurements are—

1. Base 8 yds. 1 ft., height 5 yds. 2 ft.
2. Base 14 ch. 15 lks., height 12 ch. 24 lks.
3. Base 11 ft. 4 in., height 10 ft. 5 in.
4. Base 4068 lks., height 2010 lks.
5. If the area of a triangle is 300 sq. ft., and the perpendicular height 8 ft., find the base.
6. If the area of a triangular field is 56 ac. 2 ro. 31 po., and the base measures 35 ch. 68 lks., find the perpendicular height in chains, etc.

Find the areas of the triangles whose sides are—

7. 12, 15, and 18.
8. 64, 64, and 32.
9. 1000, 1100 and 1200.
10. ·9, 1, and 1·1.
11. The base of a triangular field is 1166 lks., and the perpendicular height is 738 lks.; the field is let for £24 a year: what is the rent per acre?
12. A triangle of which the three sides are 3161, 3111, and 560 is equal in area to an isosceles triangle of which the altitude is 1220: find the base of the isosceles triangle.
13. The base of a triangle is 48 ft., the height 20 ft., and one of the sides 24 ft.: find the other side.
14. The perimeter of an equilateral triangle being 27 yds., find the area.
15. If the area of a triangular field is 1 ac. 2 ro., and the perpendicular height is 750 lks., find the length of the base.
16. The three sides of a triangular field measure 500, 530, and 600 lks. respectively: what is the value of the field at £6 10*s.* per acre?
17. What would be the cost of carpeting a triangular room the sides of which measure 20 ft., 15 ft., and 17 ft. respectively, at 4*s.* 9*d.* a square yard?
18. A triangular garden has each of its sides 36 ft.: what is its area?
19. A triangular piece of ground whose sides are 500, 800, and 500 yds. respectively is let for £30: find the rent per acre.
20. Find the side of an equilateral triangle, supposing it costs as much to pave the area at 9*d.* per square foot, as to fence the three sides at 5*s.* per foot.
21. The area of a triangle is 19 ac. 3 ro. 8 po., and the perpendicular is 18 ch.: find the length of the base in chains, etc.
22. The area of a triangle is 6 ac. 2 ro. 8 po.; its perpendicular is 826 lks.: what will be the expense of making a ditch the length of the base at 2*s.* 6*d.* a perch?
23. The three sides of a triangular fish-pond measure 293, 239, and 185 yds. respectively: what did the ground cost at £185 per acre?

24. The sides of a triangular field are 10 ch., 8 ch., and 12 ch.: find the perpendicular distance of its longest side from the opposite corner.

25. The perpendicular and base of a right-angled triangle are 357 ft. and 476 ft.: find the area.

26. Compare the area of an equilateral triangle whose side is 5 ft. long with the area of a square whose diagonal is 5 ft.

IX.—ON IRREGULAR QUADRILATERALS.

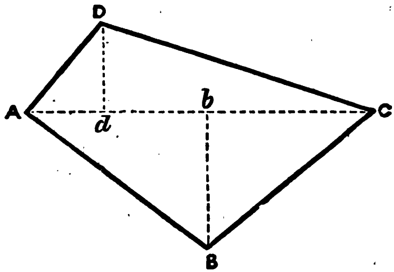
A *quadrilateral* is a figure bounded by four straight lines. An irregular quadrilateral is also called a *trapezium*

A *trapezoid* is a quadrilateral having two sides parallel.

(a) To find the area of any quadrilateral, the diagonal and perpendiculars on it from the opposite corners being given.

Let ABCD be a trapezium having diagonal AC and perpendiculars on it from opposite corners Dd and Bb.

Then the area of the quadrilateral ABCD equals the sum of the areas of the triangles ABC and ADC.



Area of triangle ADC

$$= \frac{AC \times Dd}{2}$$

Area of triangle ABC

$$= \frac{AC \times Bb}{2}$$

∴ Area of trapezium ABCD

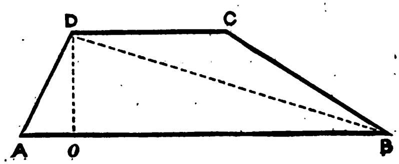
$$= \frac{AC}{2}(Dd + Bb)$$

RULE.—**Multiply the sum of the two perpendiculars by one-half the diagonal**

(b) To find the area of a trapezoid, the lengths of the two parallel sides and the perpendicular distance between them being given.

Let ABCD be a trapezoid having CB the diagonal and Do the perpendicular between the two parallel sides AB and DC.

Then the area of the trapezoid ABCD equals the sum of the areas of the triangles ADB and DCB.



$$\text{Area of triangle ADB} = \frac{AB \times Do}{2}$$

$$\text{Area of triangle DCB} = \frac{CD \times Do}{2}$$

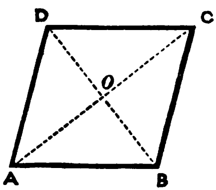
$$\therefore \text{Area of trapezoid ABCD} = \frac{Do}{2}(AB + CD)$$

RULE.—**Multiply the sum of the two parallel sides by one-half the perpendicular distance between them.**

(*c*) To find the area of a quadrilateral when the four sides and one diagonal are given.

The diagonal divides the quadrilateral into two triangles, the areas of each of which can be found according to rule (Sect. VIII. (*d*)). The sum of the areas of these triangles will give the area of the quadrilateral.

(*d*) To find the area of a quadrilateral, when the lengths of the two diagonals intersecting at right angles are given.



Let ABCD be a quadrilateral whose diagonals AC and BD intersect at *o* at right angles.

Then either of these diagonals may be regarded as the sum of the perpendiculars from the opposite corners to the other diagonal.

$$\therefore \text{Area ABCD} = \frac{AC}{2}(Do + Bo) = \frac{AC \times BD}{2}$$

RULE.—**Take one-half of the product of the two diagonals.**

EXERCISE IX.

1. In a quadrilateral ABCD, the angle at A is a right angle the sides AD and BC are parallel; AD is 76 ft., BC 30 ft., and the diagonal BD 104 ft.: find the area.

2. The area of a quadrilateral is $171\frac{1}{8}$ sq. in., the perpendiculars on a diagonal from the angles which it subtends being 9 in. and $9\frac{1}{2}$ in. respectively: what is the length of the diagonal?

3. Find the area of a trapezoid the parallel sides of which are 25 in. and 15 in., and the perpendicular distance between them 1 ft.

4. Find the area of a trapezium ABCD, of which the sides are

AB 28 yds., BC 45 yds., CD 51 yds., and DA 52 yds., and the diagonal AC = 53 yds.

5. Find the area of a trapezoid whose parallel sides are 110 ft. and 95 ft., and the perpendicular between them 400 ft.

6. Find the area of a field in the form of a trapezoid, the parallel sides being 300 lks. and 240 lks., and the perpendicular distance between them 120 lks.

7. Find the area of a trapezium whose diagonal is 115 ft., and perpendicular on it from the opposite corners 84 ft. and 25 ft. respectively.

8. Find the cost of a piece of ground, in the form of a quadrilateral whose diagonal is 2 ch. 50 lks., and perpendiculars on it from the opposite corners 1 ch. 20 lks. and 1 ch. 80 lks., at 1*s.* 4*d.* per square yard.

9. ABCD is a trapezium, of which the diagonal AC is 325 yds., AB = 123 yds., BC = 208 yds., CD = 116 yds., AD = 231 yds.: find the area in acres, etc.

10. Two sides of a garden are parallel to each other, and measure 25 ft. and 17 ft. respectively; the shortest distance between them is 27 ft.: what will the rent be at 9*d.* per square yard?

11. The parallel sides of a piece of ground measure 856 lks. and 684 lks., and their perpendicular distance is 985 lks.: find the area.

12. If the parallel sides of a garden are $65\frac{1}{2}$ ft. and $49\frac{1}{4}$ ft., and the perpendicular distance between them $56\frac{3}{4}$ ft., what did it cost, at £325 10*s.* per acre?

13. The sides of a trapezium are 335, 426, 387, and 321 yds. respectively, and the angle contained by the first two sides is a right angle: find the area.

14. ABCD is a quadrilateral; AB = 48 ch., BC = 20 ch., and the diagonal AC = 52 ch., and the perpendicular from D on AC = 30 ch.: find the area.

15. ABCD is a trapezium; BC is parallel to AD, and AB, BC, and CD each equal 325 ft., and AD = 733 ft.: find the area.

16. The diagonals of a rhombus are 88 ft. and 234 ft. respectively: find the length of the side.

17. The area of a rhombus is 354,144 sq. ft., and the diagonal is 672 ft.: find the length of the side.

18. ABCD is a field such that straight lines joining opposite corners meet at right angles at F, and the lines FA, FB, FC, FD, measure 83, 97, 125, and 228 yds. respectively: find the area of the field in acres, etc.

19. The sides of a quadrilateral field are as follows: AB = 200 lks., BC = 650 lks., CD = 905 lks., and AD = 570 lks., and the diagonal AC = 800 lks.: find the area of the field.

20. Find the area of a trapezoid whose parallel sides are 72 ft. and $38\frac{3}{8}$ ft., the other sides being 20 ft. and $26\frac{3}{8}$ ft.

X.—ON POLYGONS.

A *polygon* is a figure bounded by more than four straight lines.

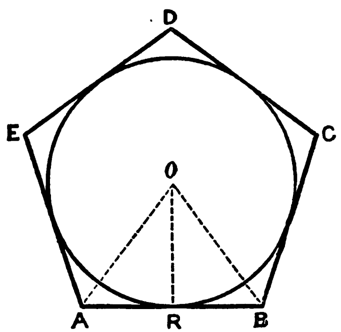
A *regular polygon* is one whose sides and angles are all equal.

A *regular pentagon* is a regular polygon having five equal sides and angles.

And so, hexagon, heptagon, octagon, nonagon, decagon, undecagon, and duodecagon are figures respectively with 6, 7, 8, 9, 10, 11, and 12 sides.

(*a*) To find the area of any regular polygon when the side of the polygon and the radius of the inscribed circle are given.

Let ABCDE be a regular polygon, having *o* the centre of the inscribed circle and *o*R the radius.



Divide it into as many triangles as it has sides, by drawing lines from the centre *o* to each of the angles.

Let n = number of sides, and r = radius of inscribed circle, and s = length of side.

Then area of each triangle

$$= \frac{AB \times oR}{2} = \frac{s \times r}{2}$$

∴ Area of all triangles

$$= \frac{n \times s \times r}{2}$$

∴ Area of polygon

$$= \frac{n \times s \times r}{2}$$

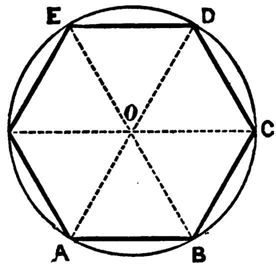
Rule.—**Multiply the perimeter of the polygon by half the radius of the inscribed circle.**

(*b*) To find the area of a regular hexagon, the length of the side being given.

Let ABCDEF be a regular hexagon, and *o* the centre of the circumscribed circle, Divide the figure into as many triangles as it has sides, namely six.

Each of these triangles is equilateral.

Let s = length of the side.



$\therefore$ Area of hexagon $= \frac{s^2\sqrt{3}}{4} \times 6 = \frac{3s^2\sqrt{3}}{2}$

RULE.—**Multiply the square of the side by one-half of the product of 3 into the square root of 3.**

NOTE.—One-half of the product of 3 into the square root of $3 = \frac{1}{2} \times 3 \times 1{\cdot}732 = 2{\cdot}598$.

(*c*) To find the length of the side of an equilateral triangle inscribed in a circle whose radius is given.

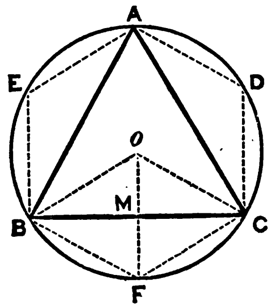
Let ABC be an equilateral triangle inscribed in the circle whose centre is *o*.

Take D, E, F, the centres of the arcs AC, AB, and BC.

Join AD, DC, CF, FB, BE, and EA.

Then AEBFCD is a regular hexagon.

Join *o*B, *o*C, and *o*F. Let *o*F cut BC in M.



Then $oM = MF = \frac{oF}{2}$

$= \frac{oC}{2}$ (Euc. I.)

$$\therefore MC = \sqrt{oC^2 - oM^2} = \sqrt{oC^2 - \frac{oC^2}{4}} = \sqrt{\frac{3oC^2}{4}} = \tfrac{1}{2}oC\sqrt{3}$$

$$\therefore BC = 2MC = 2 \times \tfrac{1}{2}oC\sqrt{3} = oC\sqrt{3}$$

RULE.—**Multiply the radius of the circle by the square root of 3.**

(*d*) To find the area of an equilateral triangle inscribed in a circle whose radius is given.

In the above figure let ABC be the equilateral triangle, *o* the centre of the inscribed circle, and *o*M the radius of the inscribed circle, and *o*C of the circumscribed circle.

Then area of equilateral triangle ABC = perimeter $\times \frac{oM}{2}$ (Sect. X. (*a*).)

But perimeter $= 3oC\sqrt{3}$ (Sect. X. (*c*).)

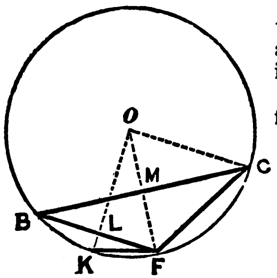
and $\frac{oM}{2} = \frac{oC}{4}$

$\therefore$ Area of ABC $= 3oC\sqrt{3} \times \frac{oC}{4} = \frac{3oC^2}{4}\sqrt{3}$

RULE.—**Multiply the square of the radius of the circumscribed circle by three-fourths of the square root of 3.**

NOTE.—Three-fourths of the square root of 3 $= \frac{3}{4}$ of $1{\cdot}732 = 1{\cdot}299$.

(*e*) To find the side of a regular duodecagon inscribed in a circle whose radius is given.



Let BC be the side of an equilateral triangle, BF the side of a regular hexagon, and FK the side of a regular duodecagon, inscribed in the circle BDFC.

Let oC be given, it is required to find FK.

Now $FB = oC$

And $FL = \frac{FB}{2} = \frac{oC}{2}$ (Euc. I.)

Also $oL^2 = oF^2 - FL^2$

$= oC^2 - \frac{oC^2}{4} = \frac{3oC^2}{4}$

$\therefore oL = \frac{oC\sqrt{3}}{2}$

Again $LK = oK - oL = oC - \frac{oC\sqrt{3}}{2} = oC\left(1 - \frac{\sqrt{3}}{2}\right) = oC\left(\frac{2-\sqrt{3}}{2}\right)$

But $FK = \sqrt{FL^2 + LK^2} = \sqrt{\frac{oC^2}{4} + \frac{oC^2(2-\sqrt{3})^2}{4}}$

$= \frac{oC}{2}\sqrt{1 + (2-\sqrt{3})^2} = \frac{oC}{2}\sqrt{1 + 4 - 4\sqrt{3} + 3}$

$= \frac{oC}{2}\sqrt{8 - 4\sqrt{3}} = \frac{oC}{2}\sqrt{4(2-\sqrt{3})} = \frac{2oC}{2}\sqrt{2-\sqrt{3}}$

$= oC\sqrt{2-\sqrt{3}}$

RULE.—**Multiply the radius of the circumscribed circle by the square root of $(2 - \sqrt{3})$.**

NOTE.—The square root of $(2 - \sqrt{3}) = \sqrt{2 - 1{\cdot}732} = \sqrt{{\cdot}268} = {\cdot}5177$.

EXERCISE X.

1. What is the area of a regular hexagon whose side is 9 in.?

2. What is the area of a regular pentagon whose side measures 92 ft. 6 in., and the radius of the inscribed circle is 63 ft. 8 in.?

3. What is the area of a regular heptagon described about a circle of 9 ft. radius whose side measures $8\frac{3}{4}$ ft.?

4. What is the area of a regular hexagon of which each side is 30 ft.?

5. What will the paving of the floor of an octagonal room with marble cost at 4*s.* 6*d.* per square foot, each side of which measures 9 ft. 6 in., and the nearest distance from one of its sides to the opposite side being 22 ft. 11 in.?

6. The radius of the circle is 1 ft.: find the area of a regular polygon of eight sides inscribed in the circle.

7. Find the perimeter of a duodecagon inscribed in a circle whose radius is 2 ft.

8. Find the perimeter of an equilateral triangle inscribed in a circle whose radius is 2 ft.

9. Compare the perimeter of an hexagon with that of a duodecagon inscribed in a circle whose radius is 3 ft.

10. Find the area of a regular hexagon inscribed in a circle whose radius is 50 ft.

11. What is the length of the side of a regular duodecagon inscribed in a circle whose radius is 10 ft.?

12. What is the area of an equilateral triangle inscribed in a circle whose radius is 10 ft.?

13. What is the length of the side of an equilateral triangle inscribed in a circle whose radius is 5 ft.?

XI.—ON IRREGULAR RECTILINEAL FIGURES.

To find the area of any irregular rectilineal figure.

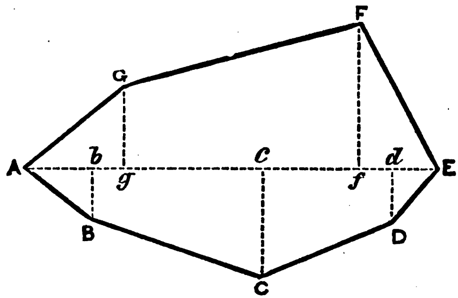
Let ABCDEFG be any irregular figure.

It may be measured by taking a diagonal AE. This line is called a *base line.* From this base line take perpendiculars to the corners, *b*B, *g*G, *c*C, *f*F, *d*D. These are called *offsets.*

These measurements are entered by surveyors in a *Field Book,* and are taken by the chain and entered in links.

The Field Book is arranged in three columns. In the middle column, commencing from the bottom, are entered the distances from the starting-station, measured on the base line of those points from which offsets are taken. In the right and left hand columns respectively are entered the right and left hand offsets.

The Field Book is used to record, besides these measured



lengths, various other particulars which may be useful in drawing a plan of any field or estate.

In the above figure, if it be taken to represent a field, the entries would be made in the Field Book thus:

	To E.	
	700	
	600	150 to D
To F 350	500	
	350	250 to C
To G 250	150	
	100	200 to B
	From A	go east.

The area would then be found thus:

Area of triangle $AgG = \dfrac{150 \times 250}{2} = 18750$ sq. lks.

" trapezoid $gfFG = (250 + 350) \times \dfrac{350}{2} = 105000$ "

" triangle $FfE = \dfrac{350 \times 200}{2} = 35000$ "

" triangle $AbB = \dfrac{100 \times 200}{2} = 10000$ "

" trapezoid $bBCc = (200 + 250) \times \dfrac{250}{2} = 56250$ "

Carried forward ... 225000

Brought forward ... 225000 sq. lks.

Area of trapezoid $cCdD = (250 + 150) \times \frac{250}{2} =$ 50000 „

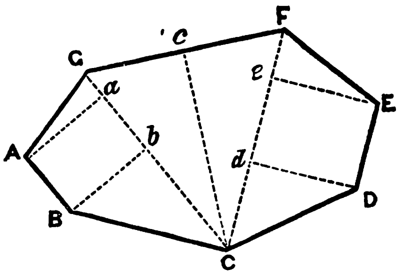
„ triangle dDE $= \frac{100 \times 150}{2} =$ 7500 „

Total 282500 „

$$282500 \text{ sq. lks.} = 2{\cdot}825 \text{ ac.}$$
$$\begin{array}{r} 4 \\ \hline 3{\cdot}300 \text{ ro.} \\ 40 \\ \hline 12{\cdot}0 \text{ po.} \end{array}$$

Ans. 2 ac. 3 ro. 12 po.

The same irregular figure or field might have been measured in a different manner, by taking three base lines, GF, FC, and CG, and at c in base line GF taking an offset to C, and at e and d respectively in base line FC taking offsets to E and D, and similarly in base line CG taking, at b and a, offsets to B and A.



If great accuracy is required in the measurement of a field, two different surveys are frequently made, when one serves to check the other.

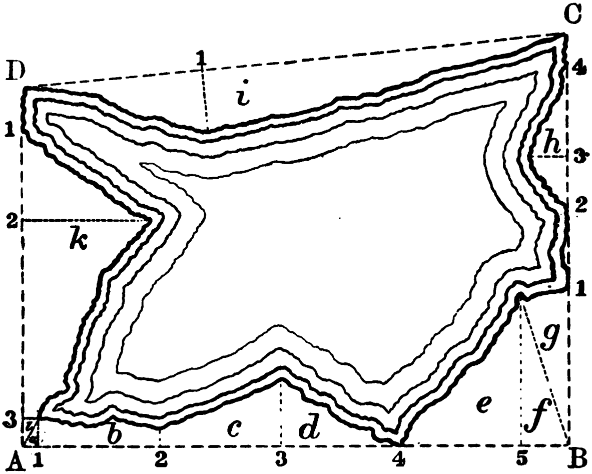
Frequently in actual land-surveying the areas of lakes or woods are required, when it is often impracticable to take base lines through the area to be surveyed. Then base lines must be taken surrounding the area, and insets taken from these base lines to the various points in the boundaries. It will be advisable to surround the area, as it were, with base lines forming a trapezoid.

Suppose the figure on page 30 to represent a mere whose area is required.

Take base lines AB, BC, CD, and DA. Let AD and BC be perpendicular to AB. Thus a trapezoid is formed. From the area of this trapezoid the necessary deductions must be made for the insets measured on the various base lines.

On base lines AB, insets will have to be taken at 1, 2, 3, 4, and 5. At 4 the entry in the Field Book would be 0, as it touches the base line.

On base line BC, insets would be taken at 1, 2, 3, and 4. The entries in the Field Book at 2 and 4 would be 0, as here again the boundary touches the base line.



On base line CD, only one inset is needed at 1.

On base line DA, at 1, in the Field Book, 0 would be entered, and insets would be needed at 2 and 3.

The Field Book would be entered thus:

	To A	
	750	
— (3) 50	650	
— (2) 250	300	
— (1) 0	100	
	From D	right angles to AB
	To D	
	1050	
— (1) 150	800	
	From C	
	To C	
	900	
(4) 0	750	
— (3) 75	600	
(2) 0	400	
— (1) 100	250	
	From B	right angles to AB

	To B	
	950	
— (5) 250	850	
(4) 0	650	
— (3) 200	450	
— (2) 75	250	
— (1) 100	50	
	From A	go east.

The area would then be found thus:

Area of trapezoid ABCD = $(750 + 900) \times \frac{950}{2}$ = 783750 sq. lks.

Deductions on base line AB.

Triangle (*a*) $\frac{50 \times 100}{2} = 2500$

Trapezoid (*b*) $\frac{175 \times 200}{2} = 17500$

" (*c*) $\frac{275 \times 200}{2} = 27500$

Triangle (*d*) $\frac{200 \times 200}{2} = 20000$

" (*e*) $\frac{200 \times 250}{2} = 25000$

" (*f*) $\frac{100 \times 250}{2} = 12500$ — 105000

Deductions on base line BC.

Triangle (*g*) $\frac{250 \times 100}{2} = 12500$

" (*h*) $\frac{350 \times 75}{2} = 13125$ — 25625

Deductions on base line CD.

Triangle (*i*) $\frac{1050 \times 150}{2} =$ 78750

Deductions on base line DA.

Triangle (*k*) $\frac{550 \times 250}{2} = 68750$

Triangle (*l*) $\frac{100 \times 50}{2} = 2500$ — 71250

280625 sq. lks.

Area = 503125 "

503125 sq. lks. = 5·03125 ac.
4
·12500 ro.
40
5·000 po.
Ans. 5 ac. 0 ro. 5 po.

EXERCISE XI.

1. Find the acreage of a field ABCDE, the measurements of which are, AB = 280 lks., BC = 450 lks., CD = 270 lks., DE = 250 lks., BG (a perpendicular on ED) = 380 lks., and AC (being parallel to ED) = 630 lks.

Draw the plans and find the areas of the fields whose measurements are given in the following Field Books:—

2.

	To B	
	600	
To G 200	560	
To F 150	480	
	470	150 to E
To D 100	380	
	100	200 to C
	From A	

3.

	To B	
	1278	
To E 116	1094	
	944	200 to D
To F 90	764	
	544	154 to C
To G 200	388	
To H 352	248	
	From A	

4.

	To D	
	538	
To C 66	400	
	From B	on left
	To B	
	629	
To D 295	179	
	From A	go east

5.

	915	144
200	729	
	670	100
236	543	
	460	183
192	400	
	262	115
304	200	
220	0	172
	From	

6.

	To B	
	2243	
	2000	
To C 728	1785	
	1000	
	560	624 to D
	From A	

	To B	
60	380	20
4	260	100
4	180	7
60	100	10
20	80	60
	0	50
	From A	

8.

	To A	
	250	
50	200	
	From C	
	To C	
	390	
	200	40
	100	30
	0	10
	From B	
	To B	
	560	
	100	30
	0	
	From A	go N.W.

9.

	To E	
	1125	
To D 260	750	
	625	250 to C
To B 230	300	
	From A	

10.

	To B	
70	959	
24	785	100
68	676	
66	564	
	508	160
95	463	
0	350	
	232	130
55	132	
	From A	

11.

	To B	
	1314	126
234	1005	
	980	52
	785	125
312	700	
	555	152
215	460	
	335	100
336	260	
360	0	232
	From A	go east

12.

	To G	
	600	80 to H
To F 140	560	
To E 150	480	
	470	200 to D
To C 100	380	
	100	150 to B
	From A	

13.

	To B	
	600	
To G 140	560	
To F 150	480	
	470	170 to E
To D 50	380	
	100	150 to C
	From A	

14.

	To D	
0	750	80 to E
To C 360	680	
	350	420 to F
To B 250	200	
0	000	0
	From A	

15. Find the area from the following Field Book of a wood whose measurements must be taken entirely from the outside.

	To A	
	300	
	From D	go at right angles to CD
	To D	
	800	
To *k* 0	550	
To *i* 100	450	
To *h* 0	350	
	From C	go at right angles to CB
	To C	
	750	
To *g* 50	500	
To *f* 0	300	
To *e* 100	200	
	From B	go north
	To B	
0	900	
To *d* 200	750	
To *c* 0	600	
To *b* 0	450	
a 150	350	
	From A	go E.S.E.

16. ABCDE is a five-sided field, and the angles at B, C, and D are right angles. AB = 20 ft., BC = 18 ft., CD = 32 ft., and DE = 13 ft.: find the length of the remaining side and the area of the field.

17. ABCD is a quadrilateral field. AB measures 48 ch., BC = 20 ch., AC = 52 ch., and the perpendicular from D on AC = 30 ch.: find the acreage of the field.

18. In a five-sided enclosure ABCDE, the following measurements are taken in chains: AB = 14, BC = 7, CD = 10, DE = 12, EA = 5, AC = 17; the angle at E is a right angle: find the area of the enclosure.

XII. ON SIMILAR RECTILINEAL FIGURES.

(*a*) Triangles.

Triangles are similar when—

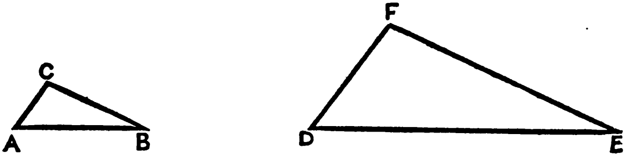
(i.) They are equiangular; or,

(ii.) They have their sides proportional.

Let ABC and DEF be two similar triangles.

Then AB : BC :: DE : EF.

And AC : AB :: FD : DE, etc. (Euc. VI.)



If, therefore, two sides of one triangle be given, and one corresponding side of a similar triangle, the other corresponding side can be found.

Let AB = 5 in., BC = 4 in., and DE = 8 in.

$$\text{Then } 5 : 4 :: 8 : \text{EF} \qquad \therefore \text{EF} = \frac{8 \times 4}{5} = 6\tfrac{2}{5} \text{ in.}$$

Let ABC and DEF be two similar triangles.

Then area ABC : area DEF :: $AB^2 : DE^2$

Or " " :: $AC^2 : DF^2$

Or " " :: $BC^2 : EF^2$

That is, areas of similar triangles are proportional to the squares of like sides.

Thus, if area ABC = 27 sq. in., and AB = 15 in., and DE = 24 in., the area of DEF can be found.

For $15^2 : 24^2$ :: 27 sq. ins. :: area DEF.

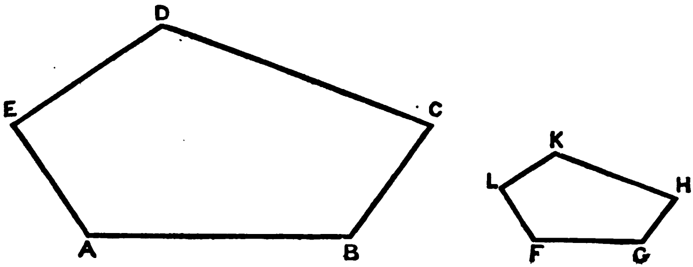
$$\therefore \text{Area DEF} = \frac{27 \times 24 \times 24}{15 \times 15} = \frac{27 \times 8 \times 8}{25} = 69{\cdot}12 \text{ sq. in.}$$

(*b*) Rectilineal figures and polygons.

Polygons and rectilineal figures are similar when—

(i.) They are equiangular; and,

(ii.) They have their sides proportional.



Let ABCDE and FGHKL be two similar rectilineal figures.

Then AB : BC :: FG : GH

And BC : CD :: GH : HK, etc. (Euc. VI.)

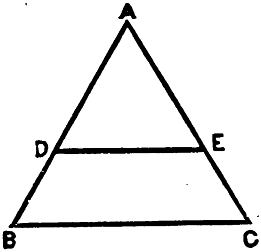
Also area ABCDE : area FGHKL :: $AB^2 : FG^2$

Or " " :: $BC^2 : GH^2$

Or as the squares of any two corresponding sides. (Euc. VI.)

Example.—The base of an isosceles triangle is 140: find the length of a line parallel to the base which will divide the triangle into two equal parts.

Let ABC be an isosceles triangle, and DE be the line parallel to the base BC, dividing the triangle into two equal parts.



It is required to find the length of DE.

Area of ADE = ½ area ABC.

But area of ABC : area of ADE :: $BC^2 : DE^2$.

$\therefore 2 : 1 :: 140^2 : DE^2$

$$\therefore DE^2 = \frac{140^2}{2}$$

$$\therefore DE = \frac{140}{\sqrt{2}} = \frac{140\sqrt{2}}{2} = 70\sqrt{2} \quad \text{Ans.}$$

This shows that a line drawn parallel to the base of a triangle equal to half the base multiplied by the square root of 2, divides the triangle into two equal parts.

EXERCISE XII.

1. The sides of a triangle are 42, 40, and 37: find the sides of another triangle similar to the first, but four times the area.

2. Divide a triangle ABC by a line DE parallel to AC into two parts proportional to the numbers 8 : 9.

3. One of the sides of a field is 420 lks.: compare its surface with that of a similar field whose corresponding side is 280 lks.

4. A triangle whose base is 156 ft. is to be divided into two parts by a line parallel to the base, so that the upper part shall be to the lower as 2 : 3: what will be the length of the line?

5. One side of a field of 40 ac. measures 605 lks., and is represented in a plan by a line 2½ in. long: what surface is occupied by the plan?

6. Divide a triangle whose base is 27 yds. into three equal parts by lines drawn parallel to the base: what must be the length of the lines?

7. What is the scale to which the plan is drawn if 1 sq. in. represents 1 sq. yd.?

8. What is the scale to which a plan is drawn when 1 sq. ft. represents 10 ac.?

9. The sides of a rectangle are as 2 : 3, and the area is 210 sq. ft.: find the sides.

10. An upright staff of 5 ft. in length casts a shadow of 4 ft.: find the height of a church spire whose shadow is 92 ft.

11. The sides of a triangle are 9, 10, and 11: find the area of a triangle cut off by a line parallel to the longest side and equal to one-third of it.

12. A field containing 20 ac. is represented on a plan by 1 sq. in.: determine the scale of the plan.

13. The area of a quadrilateral is $2140\frac{1}{2}$ sq. ft., and one of its diagonals is 63 ft.: find the area of a similar quadrilateral in which the corresponding diagonal is 61 ft.

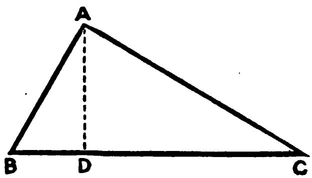
14. A square and a regular hexagon have the same perimeter: compare their areas.

15. Suppose a map of England, according to a scale of 20 miles to an inch, occupied a square foot: what space would it occupy allowing 25 miles to an inch?

XIII.—SOLUTIONS OF A FEW PROBLEMS.

PROBLEM I.—Given the three sides of a triangle 24 ft., 16 ft., and 10 ft., it is required to find (i.) the perpendicular on the base from the opposite angle, (ii.) the segments of the base, (iii.) the area of the triangle.

Let ABC be the triangle, having BC = 24 ft., AC = 16 ft., and AB = 10 ft.



Draw AD, the perpendicular from A on BC.

It is required to find (i.) AD, (ii.) BD and DC, (iii.) area of ABC.

Let BD = x. Then DC = $24 - x$

$\because AD^2 = 10^2 - x^2$

Also $AD^2 = 16^2 - (24 - x)^2$

$\therefore 10^2 - x^2 = 16^2 - (24 - x)^2$

$\therefore 100 - x^2 = 256 - 576 + 48x - x^2$

$\therefore 48x = 100 - 256 + 576 = 420$

$\therefore x = 8{\cdot}75$

$\therefore$ BD = 8·75 and DC = 24 − 8·75 = 15·25

Segments BD and DC = 8·75, 15·25 Ans.

Again, $AD^2 = 10^2 - 8{\cdot}75^2 = (10 + 8{\cdot}75)(10 - 8{\cdot}75)$

$= 18{\cdot}75 \times 1{\cdot}25 = 23{\cdot}4375$ Ans.

$\therefore AD = \sqrt{25{\cdot}9375} = 4{\cdot}841$ Ans.

Again, area of ABC $= \dfrac{24 \times 4{\cdot}841}{2} = 58{\cdot}092$ sq. ft. Ans.

Problem II.—To divide a line into extreme and mean ratio.

Let AB be the line.

It is required to find a point C so that AB : AC : : AC : CB.

Let AB $= a$. Suppose AC $= x$. Then CB $= a - x$.

$$\text{Then } a : x :: x : a - x$$

$$\therefore x^2 = a^2 - ax$$

$$\therefore x^2 + ax + \left(\frac{a}{2}\right)^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$$

$$\therefore x + \frac{a}{2} = \pm \frac{a}{2}\sqrt{5}$$

$$\therefore x = \pm \frac{a}{2}\sqrt{5} - \frac{a}{2} = \frac{a}{2}(\sqrt{5} - 1)$$

Suppose it is required to divide a line 12 in. long into extreme and mean ratio.

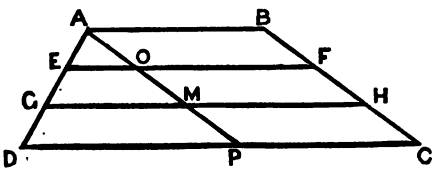
$$\text{AC} = \tfrac{12}{2}(\sqrt{5} - 1) = 6(2{\cdot}236 - 1) = 7{\cdot}416$$

$$\text{BC} = 12 - 7{\cdot}416 = 4{\cdot}584$$

Ans. 7·416 in. and 4·584 in.

Problem III.—The parallel sides of a trapezoid are respectively 8 ft. and 14 ft.; two straight lines are drawn across the figure parallel to these, so that the four parallels are equidistant: find the lengths of the two straight lines.

Let ABCD be the trapezoid, having AB = 8 ft., and DC = 14 ft.



Draw EF and GH parallel to AB and CD and equidistant.

Draw AP parallel to BC.

It is required to find EF and GH.

Now AO, OM, and MP are equal.

Because AB = 8 $\quad\therefore$ PC = 8 $\quad\therefore$ DP = 6

Now AO : AP : : EO : DP

$\therefore$ 1 : 3 : : EO : 6 $\quad\therefore$ EO = 2

But EF = EO + OF = 2 + 8 = 10 ft.

Again, AM : AP : : GM : DP

$\therefore$ 2 : 3 : : GM : 6 $\quad\therefore$ GM = 4

But GH = GM + MH = 4 + 8 = 12 ft.

Ans. 10 ft. and 12 ft.

Problem IV.—The perpendicular from the right angle of a right-angled triangle on the hypothenuse divides the triangle into two parts, one of which is double the other: show that the perpendicular divides the hypothenuse in the ratio of 1 : 2.

Let p = perpendicular on the hypothenuse dividing it into two parts, a and b.

Let a = base of smaller triangle.

Let b = base of larger triangle.

$$\text{Area of smaller triangle} = \frac{p \times a}{2}$$

$$\text{Area of larger triangle} = \frac{p \times b}{2}$$

But area of larger triangle = twice the area of smaller triangle.

$$\therefore \frac{p \times b}{2} = \frac{p \times a}{2} \times 2$$

$$\therefore \frac{p \times b}{2} = p \times a$$

$$\therefore b = 2a$$

$$\therefore a : b :: 1 : 2 \quad \text{Ans.}$$

XIV.—MISCELLANEOUS EXERCISES. (*See* EASY EXERCISES IV., A AND B.)

I.

1. Find the area of a rectangular field whose sides are 50¾ yds. and 123 yds.

2. A field is in the form of a trapezoid; its parallel sides are 10 ch. 30 lks., and 7 ch. 70 lks.; the perpendicular distance between them is 7 ch. 50 lks.: find the acreage.

3. What length of matting ¾ yd. wide will be required to cover a room 39 ft. 6 in. long by 25 ft. 6 in. wide? and find the cost at 4*s.* 6*d.* a yard.

4. How many plots of ground, each containing 1 ro. 24 po., are there in a piece of ground whose area is half a square mile?

5. Find the length of the side of a square whose area is 9659664 sq. yds.

6. Draw the plan and find area of—

	To D	
	875	
	760	325 to E
To C 120	680	
To G 0	420	
	280	265 to F
To B 210	150	
	From A	go north

II.

1. Find the area of a rhomboid whose base is 23 ft. 8 in., and height 16 ft. 9 in.

2. A ladder 25 ft. long stands upright against a wall: find how far the bottom of the ladder must be pulled out from the wall so as to lower the top one foot.

3. A rectangular court is 20 yds. longer than it is broad, and its area is 4524 sq. yds.: find its length and breadth.

4. The area of a triangle is 5 ac. 3 ro. 5·12 po., and its perpendicular distance measures 826 lks.: what will be the expense of making a ditch the whole length of its base at 2*s.* 6*d.* for 7 yds.?

5. How many poles of fencing are required to enclose a square park containing 832 ac. 2 ro. 25 po.?

6. A roll of carpet containing 30 yds., and which is 30 in. wide, will just cover a room 16 ft. long.: find the breadth of the room.

III.

1. The two diagonals of a rhombus are 50·4 and 37·8: find the side and the area.

2. What length of carpet $\frac{3}{4}$ yd. wide will cover a floor 24 ft. 10 in. by 16 ft. 6 in.?

3. The cost of planting an hexagonal field at £5 10*s.* an acre was £7 15*s.* 7$\frac{1}{4}$*d.*: find what would be the cost of fencing it round with iron railing at 2*s.* 4$\frac{1}{2}$*d.* a yard.

4. A rectangular garden contains 1200 sq. yds., and the length is to the breadth as 4 : 3: what will the fencing cost at 3*s.* 6*d.* a yard?

5. Find the area of a triangle whose sides are 25 ch., 20 ch., and 15 ch.

6. Draw a plan and find the area of the following field:—

	To B	
0	1310	130
230	1000	
	980	50
	780	130
310	700	
	550	150
220	460	
	330	94
330	260	
360	0	
	From A	

IV.

1. How much paper $\frac{3}{4}$ yd. wide will be required for a room 22 ft. long, 14 ft. wide, and 9 ft. high, if there be 3 windows and 2 doors each 6 ft. by 3 ft.?

2. Each side of an equilateral triangle is 1 ft.: find the height of the triangle.

3. It costs £643 10*s.* 9*d.* to level and turf a square cricket-ground at 9*d.* a square yard: what will it cost to enclose it with an iron paling at 7*s.* 6*d.* a yard?

4. The sides of two squares measure 77 yds. 1 ft. 9 in. and 7 yds. 2 ft. 4 in. respectively: find the side of a square whose area is equal to their sum.

5. A railway incline rises 100 ft. in perpendicular height to 4680 ft. of base: what is the length of the incline?

6. A panel measuring 2 ft. 8 in. by 1 ft. 6 in. costs 29*s.* per square foot: what would four such panels cost?

V.

1. The side of a square is 110 ft.: find the diagonal.

2. A room is 34 ft. long, 18½ ft. wide, and 12 ft. high: find the expense of papering the walls at 1*s.* 6*d.* per square yard.

3. A field containing one half-acre is laid down on a plan to a scale of 1 in. to 20 ft.: find how much paper the plan will cover.

4. A field in the form of a trapezium contains 16 ac. 3 ro. 8 po., the diagonal is 16 ch., and the perpendiculars are in the ratio of 14 : 10: find the perpendiculars.

5. A room measures 16 ft. by 21 ft., and is 11 ft. high. There is one door 7 ft. by 3 ft., and two windows each 8 ft. by 4 ft.: find the cost of papering with paper 2 ft. wide at 2½*d.* a yard.

6. What would be the cost of flooring a triangular space whose base is 21 ft. and perpendicular height 15 ft. at 8½*d.* per square foot?

VI.

1. A footpath goes along two adjacent sides of a rectangle, one side is 196 yds., the other 147 yds.: find the saving in distance made by proceeding along the diagonal instead of along the two sides.

2. A country is 500 miles long: find the length of a map which represents the country on a scale of one-eighth of an inch to a mile.

3. In a trapezium ABCD, if AB = 345 yds., BC = 156 yds., CD = 323 yds., and DA = 192 yds., and the diagonal AC = 438 yds.: find the area.

4. The sides of a triangle are 25 ft., 51 ft., and 74 ft.: find the sides of a similar triangle whose area is 2133⅓ sq. ft.

5. The area of a square cricket-field is 9 ac. 3 ro. 8·16 po.; a running-path 3·9 yds. wide is constructed close to the boundary of the field, at a cost of 4*d.* per square yard, and the remainder of the field is laid down in turf at the cost of 5*s.* 6*d.* per 100 sq. yds.: find the total cost of preparing the field.

6. A vessel is found to be 130 miles north-east of a port; after sailing due west for 80 miles, she is exactly north of the port: find what distance she is now from the port.

VII.

1. A room is 18 ft. 6 in. long and 11 ft. 3 in. wide: find the expense of carpeting the room, supposing the carpet to be 30 in. wide, and to cost 6*s.* per yard.

2. Each side of a rhombus is 24 ft., and one of the diagonals also is 24 ft.: find the area.

3. The plan of a field containing 5 ac. 3 ro. 20 po. covers 25 sq. in. of paper: what is the scale to which the plan has been drawn?

4. The area of a square field is 3 ac. 1 ro. 38 po. 20½ sq. yds.: find the size of a rectangular field whose length and breadth exceed a side of the square field by 390 yds. and 65 yds. respectively.

5. What would be the cost of painting the walls of a room 33½ ft. long, 24 ft. broad, and 12 ft. high, at 1½*d.* a square yard?

6. ABC is a triangle, and AD the perpendicular from A on BC: if AD = 13 ft., and the lengths of the perpendiculars from D on AB and AC be 5 ft and 10⅝ ft. respectively, find the area of the triangle.

VIII.

1. The distance between two towns is 31 miles, and the distance between them on a map is 7¾ in.: find the scale to which the map is drawn.

2. Find the cost of carpeting a room 18 ft. 9 in. by 17 ft. 6 in., width of carpet 2 ft., and price 4*s.* 9*d.* a yard.

3. Find the side of an equilateral triangle whose area cost as much paving at 9*d.* a square foot as palisading the three sides at 15*s.* a yard.

4. A square and an equilateral triangle have the same area: compare their perimeters.

5. A triangular field contains exactly one acre of land, and its perpendicular measures 40 yds.: find the length of the base.

6. Draw the plan and find the area of the following field whose measurements are—

	To B	
	1020	
To G 350	705	
	580	70 to F
	435	320 to E
To D 50	410	
	130	470 to C
	From A	

IX.

1. A room is 18 ft. 6 in. long, and 11 ft. 3 in. wide, and 10 ft. high: find the area of the four walls.

2. The sides of a triangular field are 350 yds., 440 yds., and 750 yds.; the field is let for £26 5*s.* a year: find at what price per acre the field is rented.

3. The area of a rhombus is 354144 sq. ft., and one diagonal is 672 ft.: find the other diagonal; also the length of the side.

4. Find the area of a regular hexagon, each side of which is 20 ft.

5. A ladder 50 ft. long reaches a point in a wall 30 ft. high: how far is the foot of the ladder from the bottom of the wall?

6. Draw the plan of the field whose measurements are the following, and find the area:—

	To G	
	1020	
To F 470	990	
	610	50 to E
To D 320	585	
To C 70	440	
	315	350 to B
	From A	

X.

1. A rectangular grass-plot is 160 ft. long, and 100 ft. broad; a gravel walk 4 ft. wide surrounds the grass-plot: find the area of the walk.

2. Suppose the carpet in a room 25 ft. long at 5*s.* a square yard to cost £6 5*s.*: find the breadth of the room.

3. The diagonals of a rhombus are 64 yds. and 36 yds.: find the area and cost of turfing at 4*d.* a square yard.

4. Find the side of an equilateral triangle whose area is 5 ac.

5. The difference between the diagonal and the side of a square is 5 ft.: find the length of the side.

6. Draw the plan of the following field and find the

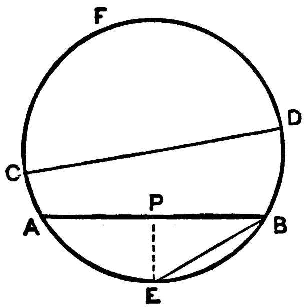
	To B	
	600	
To G 140	560	
To F 150	480	
	470	170 to E
To D 50	380	
	100	150 to C
	From A	

BOOK II.

XV.—ON THE CIRCLE.

Definitions.

A *circle* is a plane figure bounded by one line called the circumference, and is such that all lines drawn from a point within it called the centre to the circumference are equal.



A *chord* is a straight line passing through a circle terminated on both sides by the circumference, as AB.

The chord which passes through the centre is the greatest chord in a circle, and is called the *diameter*, as CD.

An *arc* is the part of the circumference cut off by a chord as AEB.

A *chord of half an arc* is a chord which bisects a given arc. Thus BE is the chord of half the arc AEB. A chord divides a circle into two segments.

A *segment* of a circle is the part of a circle bounded by a chord and the arc cut off by the chord. Thus the figures AFB and AEB are segments. The height of the arc is the perpendicular distance of the centre of the chord to the arc, as PE.

XVI.—ON THE RADIUS AND DIAMETER OF A CIRCLE.

The diameter equals twice the radius.

Let r = radius, and d = diameter.

$$\text{Then } d = 2r, \text{ and } r = \frac{d}{2}$$

EXERCISE XIII.

1. Find the radius of the circumscribed circle about a square whose side is 6 ft.

2. Find the radius of the circumscribed circle about a right-angled triangle whose base is 4 ft. and perpendicular 10 ft.

3. Find the radius of the inscribed circle in a regular hexagon whose side is 3 ft.

4. Find the radius of an inscribed circle in an equilateral triangle whose side is 4 ft.

5. Find the radius of a circumscribed circle around an equilateral triangle whose side is 4 ft.

6. The perimeter of a square is 24 ft.: find the radius of its circumscribed circle.

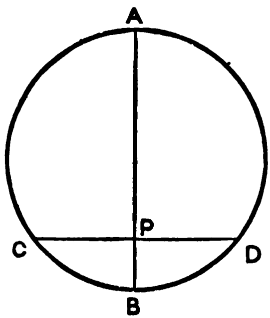
7. The diameter of a circle is 12 yds.: find the area of the inscribed square.

8. What is the radius of the circumscribed circle about a right-angled triangle whose hypothenuse is 17 ft.

9. Find the area of a square inscribed in a semicircle whose radius is 5 ft.

XVII.—ON CHORDS OF CIRCLES.

(*a*) A diameter which cuts a chord at right angles bisects the chord.



Let the diameter AB cut the chord CD at right angles in P.

Then CP = PD (Euc. III. 3.)

(*b*) If two chords cut one another, then the rectangles contained by the segments of the chords are equal.

Let the chords AB and CD cut one another in P.

Then rectangle AP.PB = rectangle CP.PD (Euc. III. 35.)

(*c*) To find the chord of an arc when the height of the segment and the diameter are given.

Let CBD be the arc of the segment, and PB = height.
It is required to find CD.

Let AB = d, PB = h

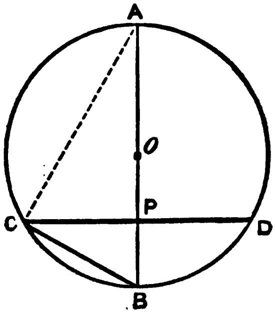
$\therefore$ AP = $d - h$

From (b) CP.PD = AP.PB $\quad\therefore$ $CP^2 = h(d-h)$

$\therefore$ CP = $\sqrt{h(d-h)}$ $\quad\therefore$ CD = 2CP = $2\sqrt{h(d-h)}$

RULE.—**From the diameter take the height of the arc, multiply the result by the height of the arc, then take twice the square root of the product.**

(d) To find the chord of half the arc when the radius of the circle and the height of the arc are given.



Let CBD be the arc, PB the height, and oB the radius.

It is required to find CB, the chord of half the arc CBD.

Let oB = r, PB = h.

$\therefore$ AB = $2r$, and AP = $2r - h$

$\therefore$ CP = $\sqrt{h(2r-h)}$

The triangle CBP is a right-angled triangle.

$\therefore$ $CB^2 = CP^2 + PB^2$

$\therefore$ CB = $\sqrt{CP^2 + PB^2}$

$\therefore$ CB = $\sqrt{h(2r-h) + h^2}$

$= \sqrt{h(2r - h + h)} = \sqrt{2rh}$

The same result may be obtained thus:

ACP and CPB are similar triangles.

$\therefore$ CB : BP :: AB : CB

$\therefore$ $CB^2 = AB \times BP$ $\quad\therefore$ CB = $\sqrt{2rh}$

RULE.—**Take the square root of twice the radius multiplied by the height of the arc.**

(e) To find the chord of half the arc when the diameter and chord of the whole arc are given.

Let the chord of whole arc CD = C, and chord of half the arc CB = c.

Then $c = \sqrt{2rh}$ $\quad\therefore$ $c^2 = 2rh$ (i.)

But $2r = d$, $h = r - o\text{P}$, and $o\text{P} = \sqrt{o\text{C}^2 - \text{CP}^2} = \sqrt{r^2 - \frac{\text{C}^2}{4}}$

$$\therefore h = r - \sqrt{r^2 - \frac{\text{C}^2}{4}}$$

$$\therefore h = \frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{\text{C}^2}{4}} = \frac{d}{2} - \tfrac{1}{2}\sqrt{d^2 - \text{C}^2}$$

$$\therefore h = \tfrac{1}{2}\{d - (\sqrt{d^2 - \text{C}^2})\}$$

Substitute these values of $2r$ and h in (i.).

$$\text{Then } c^2 = 2rh = d \times \tfrac{1}{2}\{d - (\sqrt{d^2 - C^2})\}$$

$$\therefore\ c = \sqrt{d \times \tfrac{1}{2}\{d - (\sqrt{d^2 - C^2})\}}$$

RULE.—**Take the square root of one-half the diameter multiplied by the difference between the diameter and the square root of the difference of the squares of the diameter and the chord of the whole arc.**

(*See* EASY EXERCISES VII.)

EXERCISE XIV.

1. The chord of an arc is 15 in., and the diameter of the circle is 20 in.: find the chord of half the arc.

2. The chord of an arc is 12 yds., and the chord of half the arc is $19\frac{1}{2}$ ft.: find the diameter of the circle.

3. What must be the radius of a circle in which a chord of 15 in. is 5 in. from the centre?

4. The chord of an arc is 20 ft., and the chord of half the arc is 30 ft.: find the diameter.

5. Find the radius of a circle in which two parallel chords of 6 in. and 8 in. are 1 in. apart.

6. The chord of an arc is 24 ft., and the diameter is 25 ft.: find the height of the arc.

7. In a circle whose diameter is 20 ft., a chord of 12 ft. is inscribed: find the heights of the arcs.

8. The chord of an arc is 20 ft., and the height of the arc is 4 ft.: find the diameter of the circle.

9. The chord of half an arc is 12 in., and the diameter of the circle is 36 in.: find the chord of the whole arc.

10. Find the distance from the centre of the circle of a chord whose length is 30 ft., in a circle whose radius is 20 ft.

11. The height of the arc is 12 in., and the chord of half the arc is 36 in.: find the diameter of the circle.

12. In a circle whose diameter is 5 ft. 4 in. the chord of the arc is 20 in.: find the chord of half the arc.

13. In a circle whose diameter is 540·8 the height of an arc is 20·8: what is the height of an arc double the size?

14. Find the radius of the circular arch of a bridge whose span is 120 ft., and height of arch 12·5 ft.

15. In a circle whose diameter is 113, if the chord of an arc is 15, what is the chord of twice the arc?

XVIII.—ON THE CIRCUMFERENCE OF CIRCLES.

(*a*) To find the ratio of the circumference to the diameter.

It has been shown in Section X. that the side of a regular hexagon inscribed in a circle equals the radius.

Therefore the chord of one-sixth of the circumference equals the radius.

If, then, the diameter = 2, the radius = 1, and the chord of the sixth part of the circumference = 1.

Again, it has been shown in Section XVII. (*e*) that the chord of half an arc can be obtained from the chord of the whole arc and the diameter of the circle.

If c is the chord of half the arc, and C is the chord of the whole arc,

$$c^2 = \tfrac{1}{2}d\{d - (\sqrt{d^2 - C^2})\}$$

∴ The chord of one-twelfth of the circumference when $d = 2$ can be found by making $C = 1$.

Let c_1 = chord of one-twelfth of the circumference.

$$\therefore c_1^2 = \tfrac{1}{2} \times 2\{2 - (\sqrt{4 - 1})\} = 2 - \sqrt{3}$$

$$\therefore c_1 = \sqrt{2 - \sqrt{3}} = \cdot 51763809$$

Again, the chord of one twenty-fourth of the circumference may be found by making $C = \sqrt{2 - \sqrt{3}}$

Let c_2 = chord of one twenty-fourth of the circumference.

$$\therefore c_2^2 = \tfrac{1}{2} \times 2\left\{2 - \left(\sqrt{4 - (2 - \sqrt{3})}\right)\right\} = 2 - (\sqrt{2 + \sqrt{3}})$$

$$\therefore c_2 = \sqrt{2 - \sqrt{2 + \sqrt{3}}} = \cdot 261052384$$

Again, the chord of one forty-eighth of the circumference may be found by making $C = \sqrt{2 - \sqrt{2 + \sqrt{3}}}$

Let c_3 = chord of one forty-eighth part of the circumference.

$$\therefore c_3^2 = \tfrac{1}{2} \times 2\left\{2 - \sqrt{4 - (2 - \sqrt{2 + \sqrt{3}})}\right\}$$

$$= 2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}$$

$$\therefore c_3 = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} = \cdot 130806258$$

Now, if this successive bisection of the arc be continued, we shall ultimately reach a point when the chord of the very small arc will more and more nearly equal the length of the arc itself.

By the continuation of the above method we obtain—

The chord of the 96th part of the circumference when $d = 2$ = ·06543816
„ „ 192nd „ „ „ = ·03272346
and so on.
The chord of the 12288th „ „ „ = ·0005113269
„ „ 24576th „ „ „ = ·0002556634

Thus it is seen that the chord of half the arc is gradually approaching half the arc itself.

In the last result the length of half the arc is the same as the half of the chord up to the tenth place of decimals.

Therefore the length of the circumference when $d = 2$ is 24576 × ·0002556634 = 6·2831852. If $d = 1$, the circumference will = 3·1415926, which is the ratio of the circumference to the diameter true to the sixth place of decimals.

This ratio is taken as 3·1416 : 1, which is slightly in excess of the true value, but gives a result sufficiently near for all practical purposes.

The ratio 3·1416 : 1 may also be taken as $3\frac{1}{7}$: 1. This is still more in excess, but for practical purposes gives a sufficiently correct result, as it is true to the one-thousandth part of the diameter.

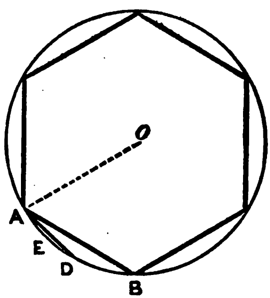
The ratio of the circumference to the diameter is represented by π.

$$\therefore \pi = \mathbf{3{\cdot}1416}, \text{ or } \tfrac{355}{113}, \text{ or } \tfrac{22}{7}$$

The results obtained above may be represented geometrically thus:

AB is the chord of the 6th part.
AD „ „ 12th „
AE „ „ 24th „

And thus it will be readily seen that at each bisection of the arc, the length of the chord more nearly approaches the length of the arc itself.



(*b*) To find the circumference when the diameter is given.

Let c = circumference, and d = diameter.

Then $c = d \times \pi$

RULE.—**Multiply the diameter by π**; *or*, **Multiply twice the radius by π.**

(c) To find the diameter, circumference given

$$d \times \pi = c \qquad \therefore d = \frac{c}{\pi} \text{ (i.)}$$

$$\text{Again } 2r \times \pi = c \qquad \therefore r = \frac{c}{2\pi} \text{ (ii.)}$$

RULES I.—**Diameter equals circumference divided by π.**
II.—**Radius equals circumference divided by 2π.**

See EASY EXERCISES V. A AND C.

EXERCISE XV.

When $\pi = 3\frac{1}{7}$, find the circumferences of circles whose diameters are—

1. 15 ft. **2.** 1 ft. 9 in. **3.** 5 yds. 1 ft. 6 in. **4.** 4 ch. 75 lks.

When $\pi = 3{\cdot}1416$, find the circumferences of circles whose radii are—

5. 20 in. **6.** 3 ft. 5 in. **7.** 450 yds. **8.** 13 ft. 6 in.

When $\pi = 3\frac{1}{7}$, find the radii of circles whose circumferences are—

9. 10 ch. **10.** 1 mile. **11.** 100 ft. **12.** 1·4 in.

When $\pi = 3{\cdot}1416$, find the diameters of circles whose circumferences are—

13. 1 ft. **14.** 1 fur. **15.** 360 ft. **16.** 4 in.

17. The difference between the circumference and diameter of a circle is 20 ft.: find the radius.

18. How often will a carriage wheel which is 3 ft. 6 in. in diameter revolve in going 1 mile?

19. The sum of the circumference and diameter of a circle is 20 ft.: find the radius.

20. Compare the perimeter of a square and that of its inscribed circle.

21. Compare the circumference of a circle with that of its inscribed hexagon.

22. A boy can walk round a circle in 50 min.: how long would he take to travel along the diameter and back again?

23. It takes 15 min. to walk round a circular plantation at the rate of 3 miles an hour: what time would it take to walk across it through the centre?

24. Two velocipedes have their driving-wheels 5 ft. and 4 ft. 6 in. in diameter respectively: suppose each has made a thousand revolutions, how far will one be ahead of the other?

25. A locomotive engine is travelling at the rate of 45 miles an hour, and the diameter of the driving-wheel is 6 ft.: how many times will it revolve in 1 min.?

26. Compare the circumference of a circle whose radius is 20 ft. with that of the inscribed duodecagon.

XIX.—ON ARCS OF CIRCLES.

(*a*) To find the angle at the centre subtended by an arc when the circumference and the length of the arc are given.

Arcs are proportional to the angles they subtend (Euc. VI. 33).

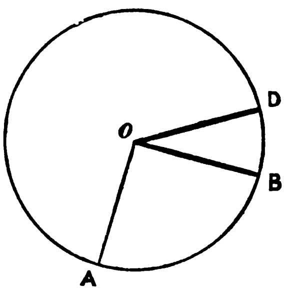
$\therefore$ Arc AB : Arc AD :: $\angle$ A*o*B : $\angle$ A*o*D

But the angles at the centre of the circle equal 4 right angles = 360°.

$\therefore$ Circumference of circle : arc AD :: 360° : $\angle$ A*o*D

$$\therefore \angle AoD = \frac{\text{Arc AD} \times 360^\circ}{\text{Circumference of circle}}$$

Rule.—**Multiply 360° by the length of the arc, and then divide by the length of the circumference.**



(*b*) To find the length of the arc when the circumference and the angle subtended by the arc at the centre are given.

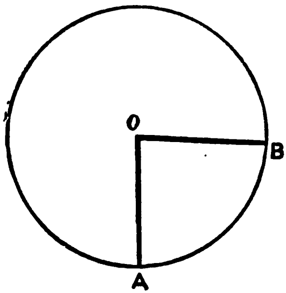
Let l = length of the arc, c = circumference, and A*o*B = the angle.

Then c : arc AB :: 360° : $\angle$ A*o*B

$$\therefore \text{Arc AB} = \frac{c \times \angle AoB}{360^\circ}$$

Rule.—**Multiply the circumference of the circle by the number of degrees in the angle, and divide by 360°.**

(*c*) To find the magnitude of an angle at the centre subtended by an arc equal to the radius.



Let A*o* = radius, and AB = arc = *o*A = *r*.

It is required to find ∠ A*o*B.

Circumference of circle : arc AB :: 360° : ∠ A*o*B.

$\therefore c : r :: 360° : \angle AoB$

$$\therefore \angle AoB = \frac{360° \times r}{c} = \frac{360° \times r}{2r\pi}$$

$$= \frac{180°}{\pi} = \frac{180°}{3·1416}$$

$$= 57°·29577$$

$$= 57° \; 17' \; 45''$$

This is the magnitude of the angle in any circle which is subtended by an arc equal to the radius, and is taken for the unit angle and used for measuring the magnitude of other angles.

It is important that the meaning of the symbol π should be clearly understood.

π always means 3·1416.

But it may refer to—

(i.) *A circle*, when it means the circumference is 3·1416 times the diameter.

(ii.) *An angle at the centre of a circle*, when it means 3·1416 times 57° 17′ 45″ = 180° = 2 right angles.

Example I.—What is the magnitude of an angle whose circular measure is $\frac{3}{4}$?

That is, what is the magnitude of an angle whose arc equals $\frac{3}{4}$ of the radius.

As $1 : \frac{3}{4} :: 57° \; 17' \; 45 : \angle$ required

$$\therefore \angle \text{ required} = \frac{57° \; 17' \; 45'' \times 3}{4} = 42° \; 51' \; 18''·75 \text{ Ans.}$$

Example II.—What is the circular measure of an angle of 75°?

Now, the circular measure of 180° = 3·1416 = π

$\therefore$ 180° : 75° :: 3·1416 : circular measure of 75°

$$\therefore \text{ Circular measure of } 75^\circ = \frac{75^\circ \times 3{\cdot}1416}{180^\circ} = \frac{5}{12} \text{ of } 3{\cdot}1416 = 1{\cdot}309 \text{ Ans.}$$

That is, the arc which an angle of 75° subtends is 1·309 times the radius of the circle.

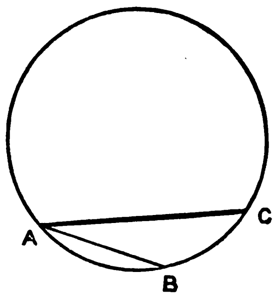
Or 57° 17′ 45″ is the angle which subtends an arc equal to the radius.

And 75° is the angle which subtends an arc 1·309 times the radius.

(*d*) The chord of an arc being given, and also the chord of half the arc, to find the length of the arc.

Let AC be the chord of the arc ABC, AB the chord of half the arc, and length of the arc = l.

$$\text{Then } l = \frac{8\text{AB} - \text{AC}}{3}$$



RULE.—**From eight times the chord of half the arc, subtract the chord of the whole arc, and take one-third.**

NOTE—This rule is not exact, but gives a very close approximation. For small angles the error is inappreciable. The proof of the rule requires a knowledge of mathematics beyond the limits of this work.

EXERCISE XVI.

1. The radius of a circle is 2 ft., and the length of the arc 15 in.: find the angle subtended at the centre by the arc.

2. The radius of a circle is 10 in., and the length of the arc 20 in.: find the angle subtended at the centre by the arc.

3. In a circle whose radius is 1 ft., find the angle at the centre subtended by an arc of 1 in.

4. Find the angles whose circular measures are—(i.) $\frac{2}{5}$, (ii.) 1, (iii.) $\frac{2}{3}$, (iv.) $\frac{9}{4}$.

5. What are the circular measures of the following angles: (i.) 45°: (ii.) 150°, (iii.) 112° 30 ?

6. The chord of an arc is 36 in., and the chord of half the arc is 19 in.: find the length of the arc.

7. The chord of an arc is 18 ft., and the chord of half the arc is 12 ft.: find the length of the arc.

8. If the chord of an arc be 24 ft., and the height of the arc 9 ft., find the length of the arc.

9. How many inches long is an arc subtending an angle of 36° at the centre of a circle, the radius of which is 2 ft. $4\frac{1}{4}$ in.?

10. Find the angle formed by the hands of a watch at 20 min. past 3 o'clock.

11. When between three and four o'clock, will the hands of a watch form an angle of 45°?

12. Find the angle at the circumference which is subtended by an arc of 10 in. in a circle whose radius is 10 in.

13. Compare the radii of two circles in which an angle of 40° at the centre of one, and an angle of 60° at the centre of the other, subtend equal arcs.

14. The chord of an arc is 6 in., and the radius of the circle 9 in.: find the length of the arc.

15. What is the length of the arc when the chord of the whole arc is 16 ft., and the height of the arc 6 ft.?

16. Find the number of degrees subtended by an arc of a circle whose diameter is 18 ft., the length of the arc being 12 ft.

17. The minute-hand of a clock makes an arc of 11 in. in ten minutes: find the radius of the face of the clock.

18. If the radius of a clock-face is 7 in., what is the length of the area travelled by the minute-hand in $5\frac{1}{2}$ minutes?

19. The chord of an arc of a circle is 8 in. long, and is 2 in. from the centre: find the angle at the centre subtended by the chord.

20. What is the circular measure of an angle in a regular pentagon?

21. What must be the radius of a driving-wheel of an engine which is to revolve 10 times in travelling one-twelfth of a mile?

22. What is the magnitude of an angle at the centre subtended by an arc equal to one-half the radius?

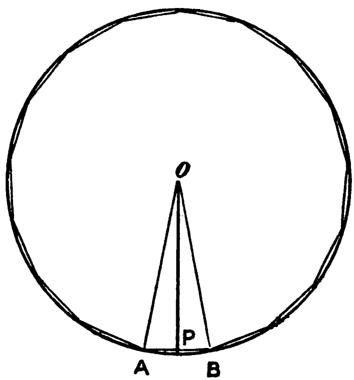
23. Two chords of a circle AB and AC are 6 in. and 8 in. long respectively; the radius of the circle is 5 in.: find the angle at the centre of which BC is the chord.

24. If the chord of the whole arc of a circle is 8 in., and of half the arc 5 in., what is the radius of the centre?

XX.—ON AREAS OF CIRCLES.

(*a*) To find the area of a circle, the circumference and radius being given.

The area of a regular polygon inscribed in a circle has been shown to be equal to the perimeter multiplied by half the radius of the inscribed circle (Sect. X. (*a*)).



Let s = side of polygon AB; n = number of sides, and r = *o*P, the radius of the inscribed circle.

Then area of polygon $= \dfrac{n \cdot s \cdot r}{2}$

If the number of sides be indefinitely increased, it follows that—

(i.) The area of the polygon will ultimately equal the area of the circle.

(ii.) The perimeter of the polygon will become the circumference of the circle; that is, $n \times s = c$.

(iii.) The radius of the inscribed circle will become the radius of the circle; that is, $oP = oA = r$.

$$\text{Hence area of circle} = \frac{c \times r}{2}$$

Rule.—**Multiply the circumference by the radius, and take one-half of the product.**

(*b*) To find the area of a circle, radius given.

$$\text{Area of circle} = \frac{c \times r}{2}$$

$$\text{But } c = 2\pi r \quad \text{(Sect. XVIII. (}b\text{).)}$$

$$\therefore \text{Area of circle} = \frac{2\pi r \times r}{2} = \pi r^2$$

Rule.—**Multiply the square of the radius by π.**

(*c*) To find the area of a circle, diameter given.

$$\text{Area of circle} = \pi r^2$$

$$\text{But } r = \frac{d}{2}$$

$$\therefore \text{Area of circle} = \frac{\pi d^2}{4}$$

RULE.—**Multiply the square of the diameter by one-fourth of π.**

NOTE.—One-fourth of π = ·7854.

(*d*) To find the area of a circle, circumference given.

$$\text{Area of circle} = \frac{\pi d^2}{4}$$

$$\text{But } d = \frac{c}{\pi}$$

$$\therefore \text{Area of circle} = \frac{\pi c^2}{4\pi^2} = \frac{c^2}{4\pi}$$

RULE.—**Divide the square of the circumference by four times π.**

NOTE.—The fraction $\frac{1}{4\pi}$ = ·07958.

Hence this rule becomes, Multiply the square of the circumference by ·07958.

(*e*) To find the radius, area being given.

$$\text{Now } \pi r^2 = \text{area}$$

$$\therefore r = \sqrt{\frac{\text{area}}{\pi}}$$

RULE.—**Divide the area by π, and then extract the square root.**

(*f*) To find the circumference, area given.

$$\text{Now } \frac{c^2}{4\pi} = \text{area}$$

$$\therefore c^2 = \text{area} \times 4\pi$$

$$\therefore c = \sqrt{\text{area} \times 4\pi} = 2\sqrt{\text{area} \times \pi}$$

RULE.—**Take twice the square root of the area multiplied by π**

(g) To find the diameter, area given.

$$\text{Now } \frac{d^2\pi}{4} = \text{area}$$

$$\therefore d^2 = \frac{\text{area} \times 4}{\pi}$$

$$\therefore d = 2\sqrt{\frac{\text{area}}{\pi}}$$

RULE.—**Take twice the square root of the area divided by π.**

(h) To find the area of a circular ring.

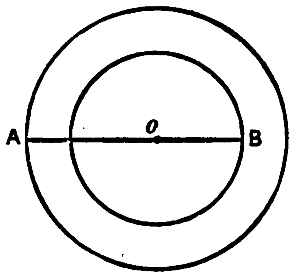
Let the radius of the larger circle $oA = R$, and the radius of the smaller circle $oB = r$.

Then area of larger circle $= \pi R^2$

Then area of smaller circle $= \pi r^2$

$\therefore$ Area of ring $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

$= \pi(R + r)(R - r)$



RULE.—**Multiply the product of the sum and difference of the inner and outer radii by π.**

(*See* EASY EXERCISES V. B, D, E, F, AND G.)

EXERCISE XVII.

When $\pi = 3\frac{1}{7}$, find the areas of the circles whose radii are—

1. 42 ft. 2. $\frac{1}{4}$ mile 3. 100 ft. 4. 80 ft.

When $\pi = 3{\cdot}1416$, find the areas of circles whose circumferences are—

5. 90 in. 6. 3500 lks. 7. $60\frac{1}{2}$ ch. 8. 7500 yds.

When $\pi = 3{\cdot}1416$, find the areas of circles whose radii are—

9. 25 ft. 10. 992 ft. 11. $\frac{1}{4}$ mile. 12. 3 ins.

13. A road runs round a circular shrubbery; the outer circumference is 500 ft., and the inner 420 ft.: find the area of the road.

14. The area of a circle is half an acre: find the circumference in feet.

15. The radius of a circle is 8 ft.: find the radius of another circle of half the area.

16. Find the expense of paving a circular court of 40 ft. in diameter at 2*s*. 3*d*. per square foot.

17. The side of a square is 18 ft.; a circle is described round it: find the area between the circle and the square.

18. The area of a square is equal to that of a circle: compare their perimeters.

19. The sides of a triangle are 30, 40, and 50: find the area of a circle having the same perimeter.

Find the radii of the circles whose areas are—

20. 1 sq. mile. 21. 2 ac. 3 ro. 10 po.

22. A cow, tethered, is allowed to feed on $2\frac{1}{2}$ ac. of ground: how many yards long is the rope by which it is tied?

23. A square field has an area of $5\frac{1}{2}$ ac.: find the area of a circle having the diagonal for its diameter.

24. A circular grass-plot contains 1760 sq. yds., and is surrounded by a gravel walk 10 ft. broad: find the area of the walk.

25. The diameter of a circle is divided in the ratio of $1\frac{3}{8} : 2\frac{1}{2}$, and circles are described on the segments as diameters: compare the area of the three circles.

26. Find the number of grass sods, each 18 in. by 12 in., that will be required to cover a circular piece of ground $15\frac{1}{2}$ ft. in diameter.

27. A semicircular piece of ground whose diameter is 15 yds. is covered with carpet 2 ft. wide: what will it cost at 3*s.* 6*d.* a yard, allowing 11·6425 yds. for waste, and taking $\pi = 3{\cdot}1416$.

28. The perimeter of an equilateral triangle is equal to the circumference of a circle: compare their areas.

29. Compare the areas of an equilateral triangle and a regular hexagon inscribed in a circle.

30. A circular lawn is 2 ac. in extent, and the road round it is one-half an acre: find the width of the road.

31. Two men bought a grindstone one yard in diameter: what part of the diameter may each man grind down?

32. A circular fountain 12 ft. in diameter touches the sides of a square plot within which it is situated: find the area of the remainder of the plot.

33. A curbstone 15 in. broad is put to a well 7 ft. in diameter at a cost of 17*s.* 6*d.*: find the price per square foot.

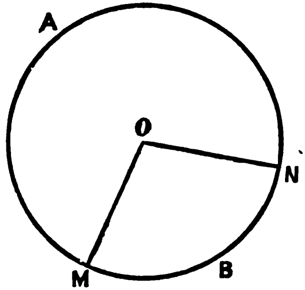
34. The paving of a semicircular area with tiles at 2*s.* 6*d.* a square foot cost £10: what was the length of the semicircular arc?

35. Find the cost of making a path 7 ft. wide round a circular ground whose perimeter is 352 yds. at 3*s.* 9*d.* per square yard ($\pi = 3\frac{1}{7}$).

XXI.—ON AREAS OF SECTORS OF CIRCLES.

A sector of a circle is a figure bounded by two radii and the arc subtended by them.

In the circle AMBN the figures M*o*NA and M*o*NB are sectors.



(*a*) To find the area of a sector, length of the arc and radius being given.

Now, area of circle : area of sector : : circumference of circle : length of the arc (Euc. VI. 33).

Let l = length of the arc, and r = radius.

Then πr^2 : area of sector : : $2\pi r : l$

$$\therefore \text{ Area of sector } = \frac{l \times \pi r^2}{2\pi r} = \frac{lr}{2}$$

Rule.—**Multiply the length of the arc by one-half of the radius.**

(*b*) To find the radius of the circle, the area of the sector and length of the arc being given.

$$\text{Now } \frac{lr}{2} = \text{area of sector}$$

$$\therefore r = \frac{\text{area of sector} \times 2}{l}$$

Rule.—**Multiply the area of the sector by two, and divide the product by the length of the arc.**

(c) To find the number of degrees subtended by the arc of the circle, the area of the sector and the radius being given.

Let N = number of degrees subtended by the arc.

Then area of circle : area of sector : : 360° : N

$\therefore \pi r^2$: area of sector : : 360° : N

$$\therefore N = \frac{\text{area of sector} \times 360^\circ}{\pi r^2}$$

Rule.—**Multiply 360° by the area of the sector, and divide the product by the area of the circle.**

(*See* Easy Exercises VI. A and B.)

EXERCISE XVIII.

1. The radius of a sector is 25 ft., and the length of the arc 8 ft.: find the area of the sector.

2. The length of an arc of a sector is 8·6 in., and the radius 3 ft. 6 in.: find the area of the sector.

3. Find the area of a sector whose radius is 8 ft., and whose arc subtends an angle of 159°.

4. Find the area of a sector whose radius is 21 ft., and the length of whose arc is 15 ft.

5. In a circle whose radius is 12 ft., what must be the length of the arc so that the area of the sector will be 100 sq. ft.?

6. The area of a sector is 357 sq. ft., the length of the arc is 96 ft.: find the radius.

7. The area of a sector is 115 sq. ft., the area of the circle is 700 sq. ft.: find the length of the arc of the sector.

8. Find the area of a sector the chord of whose arc is 6 in., and the radius of the circle 9 in.

9. What is the radius of a circle in which a sector of 75° contains 120 sq. ft.?

10. The radius of a circle is 6 ft.: find the area of the sector and the length of the arc which subtends an angle of 115°.

11. If the area of a sector be 20 sq. ft., and the radius of the circle 5 ft., find the number of degrees subtended by the arc of the sector.

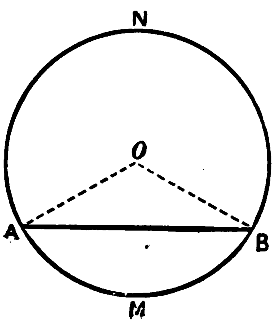
12. From each corner of an equilateral triangle whose side is 5 ft. sectors of a radius of 1 ft. are cut off: find the area of the remaining portion of the triangle.

13. At each corner of a square of 4 ft. sectors are added of 1 ft. radius: find the total area of the figure.

14. What is the area of a sector the chord of whose arc is 24 ft., and the height of the arc 6 ft.?

15. The difference between the areas of two squares inscribed and circumscribed about a circle is 338 sq. ft.: find the radius of the circle.

XXII.—ON AREAS OF SEGMENTS OF CIRCLES.



A segment of a circle is a figure bounded by a chord of a circle and that part of the circumference which the chord cuts off.

In the circle AMBN the figures ABM and ABN are segments.

(a) To find the area of a segment.

It is evident that—

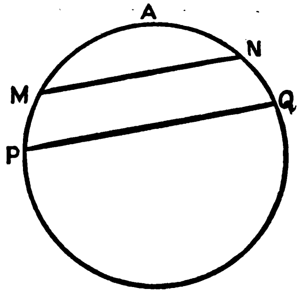
(i.) Area of segment ABM = area of sector A*o*BM − area of the triangle A*o*B.

(ii.) Area of segment ABN = area of sector A*o*BN + area of the triangle A*o*B.

RULES I.—**Area of the lesser segment equals the area of the lesser sector less the area of the triangle whose base is the chord of the segment and whose apex is the centre of the circle.**

II.—**Area of the greater segment equals the area of the greater sector plus the area of the triangle whose base is the chord of the segment and whose apex is the centre of the circle.**

(b) To find the area of a circle between two parallel chords.



Let MN and PQ be two parallel chords in the circle APN.

It is evident that the area between the two parallel chords equals the difference between the areas of the segments PAQ and MAN.

RULE.—**Take the difference of the areas of the two segments cut off by the two parallel chords.**

EXERCISE XIX.

1. Find the area of the segment of a circle whose radius is 12 ft., and the chord of whose arc is 16 ft.

2. Find the area of the segment whose chord is 22 ft., and height of segment 22 ft.

3. Find the area of the segment whose chord is 40 ft., and chord of half the arc 25 ft.

4. The radius of a circle is 20 in.: find the area of the greater segment cut off by a chord of 24 in.

5. If the chord of a segment be 48 ft., and the height of the segment 6 ft., what is the area.

6. The diameter of a circle is 30 in.: find the area of a segment cut off by a chord of 24 in.

7. The radius of a circle is 15 ft.: find the areas into which it is divided by a chord equal to the radius.

8. The radius of a circle is 12 ft.; two parallel chords are drawn on the same side of the centre, one subtending an angle of 60° at the centre, and the other an angle of 90°: find the area of the belt between the chords.

9. Find the greater segment cut off by a chord of 10 in. from a circle whose radius is 13 in.

10. Find the area between two parallel chords of 6 in. and 7 in. respectively, lying on the same side of the centre in a circle whose radius is 5 in.

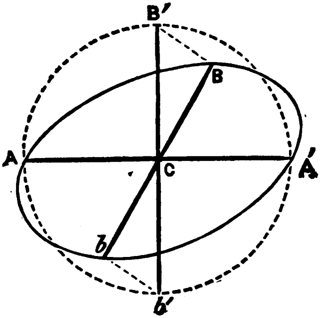
11. If the centre of a circle whose diameter is 16 be in the circumference of a circle whose diameter is 30, find the area of the figure common to both circles.

12. Two parallel chords are drawn one on each side of the diameter of a circle whose radius is 1 ft.; the chords are 9 in. and 15 in. respectively: find the area of the circle between the chords.

13. Two equal chords of 3 ft. each are drawn from a point in the circumference of a circle, and contain an angle equal to two-thirds of a right angle: find the area of the circle ($\pi = 3\frac{1}{7}$).

14. A circle is described about a square whose side is 6 yds.: what part of a square chain is the area between the square and the circle?

XXIII.—ON THE AREA OF AN ELLIPSE.



An ellipse is a figure formed by the projection of a circle on an inclined plane.

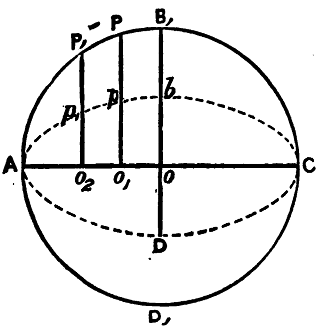
Let AB′A′b' be a circle, and ABA′b any plane passing through its centre. Let perpendiculars be drawn from every point on this circle to this plane as B′B, $b'b$. The curve formed by the feet of these perpendiculars ABA′b is an ellipse.

To find the area of an ellipse.

Let AbCD be an ellipse.

Let AC and bD be the major and minor axes.

Draw the circle on the major axis AB_1CD_1. Drop parallel ordinates B_1bo, Ppo_1, and $P_1p_1o_2$.



From the properties of the ellipse,

$Po_1 : po_1 :: P_1o_2 : p_1o_2 :: B_1o : bo :: Ao : bo.$

Let $Ao = r$, and $bo = r_1$.

If the ellipse and the circle be divided into an indefinite number of bands by ordinates such as B_1o, Po_1, etc., then the area of any one of the circular bands as Po_1oB_1 : to the area of the corresponding elliptical band $po_1ob :: B_1o : bo :: r : r_1$.

$\therefore$ Area of circle : area of ellipse $:: r : r_1$

$\therefore \pi r^2 :$ area of ellipse $:: r : r_1$

$$\therefore \text{Area of ellipse} = \frac{\pi r^2 r_1}{r} = \pi r r_1$$

Rule.—**Multiply the product of the semidiameters by π.**

(*See* Easy Exercises VIII.)

EXERCISE XX.

1. Find the area of an ellipse whose axes are 34 ft. and 30 ft.

2. Find the area of an ellipse whose axes are 30 ft. and 28 ft.

3. The length and breadth of an ellipse are 24 ft. and 18 ft.: what is the area?

4. What would be the cost of asphalting an elliptical space at 1*s.* 6*d.* a square yard, the two diameters being 28 ft. and 22 ft. respectively?

5. The cost of covering an elliptical space with carpet at 5*s.* 9*d.* a square yard was £16 17*s.* 4*d.*; the major axis measured 28 ft.: what was the length of the minor axis?

6. From an elliptical piece of paper whose diameters are 4 in. and 5 in. respectively, another ellipse is cut whose diameters are 2 in. and 3 in.: what is the area of the paper left?

7. Find the area of an ellipse whose diameters are 15·5 ft. and 35·5 ft. respectively.

8. Two equal chords, 15 in. in length, are drawn from a point in the circumference of a circle, and form an angle equal to $\frac{2}{3}$ of a right angle: find the area of the circle.

9. The area of a sector being 125 sq. ft., and of the whole circle 400 sq. ft.: find the angle of the sector.

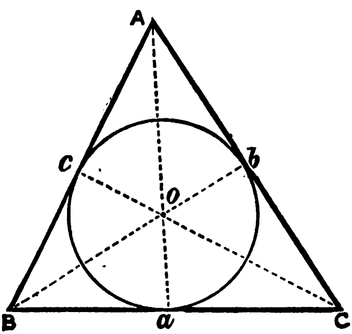
10. Compare the area of a circle whose diameter is 5 ft. with that of an ellipse whose conjugate diameters are 5 ft. and 4 ft. respectively.

11. The largest possible ellipse is described in a rectangle whose sides are 10 ft. and 8 ft.: compare the areas of the ellipse and rectangle.

12. What must be the minor axis of an ellipse so that the ellipse will be $\frac{3}{4}$ of the area of the circle described on its major diameter, which is 10 ft.?

XXIV.—SOLUTION OF A FEW PROBLEMS.

Problem I.—Given the three sides of a triangle, to find the radius of the inscribed circle.



Let ABC be the triangle, whose sides are a, b, and c.

Let oa = radius of the inscribed circle = r.

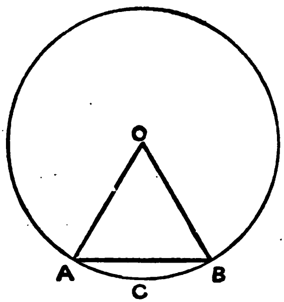
Join Ao, Bo, and Co.

Area of triangle ABC = area of triangles AoB + BoC + AoC $= \frac{c \cdot r}{2} + \frac{a \cdot r}{2} + \frac{b \cdot r}{2}$

$$= \left(\frac{a+b+c}{2}\right)r$$

$$\therefore r = \frac{\text{area of the triangle ABC}}{\frac{a+b+c}{2}} \quad \text{Ans.}$$

NOTE.—If the area of the triangle be divided by one-half of the sum of the sides, the radius of the inscribed circle will be obtained.



PROBLEM II.—The radius of a circle is 10 in., and the angle of the sector is 60°: find the area of the segment.

Area of circle $= \pi r^2 = 10^2 \times 3{\cdot}1416$
$= 314{\cdot}16$ sq. in.

Area of sector $= \dfrac{60° \times 314{\cdot}16}{360°}$
$= 52{\cdot}36$ sq. in.

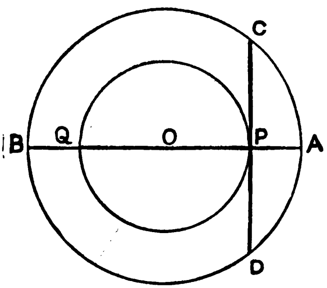
The triangle AOB is equilateral.

$\therefore$ Area of AOB $= \dfrac{10^2 \times \sqrt{3}}{4}$
$= 10^2 \times {\cdot}433 = 43{\cdot}3$ sq. in.

$\therefore$ Area of segment ABC $= 52{\cdot}36 - 43{\cdot}3$
$= 9{\cdot}06$ sq. in. Ans.

PROBLEM III.—Show that the area of a ring is equal to that of a circle whose diameter is a chord of the exterior circle touching the interior circle.

Let BO = R = radius of exterior circle.
QO = r = radius of interior circle.
Let CD = chord of circle BCAD, which is also a tangent to the inner circle.



It is required to prove that the area of the circular ring is equal to the area of a circle whose diameter is CD.

BP = BO + OP = R + r
AP = BO − OQ = R − r
$\therefore$ BP × AP = (R + r) (R − r)
But BP × AP = CP . PD = CP²
(Euc. III. 35) (Sect. XVII. (b)).
$\therefore$ (R + r) (R − r) = CP²

Multiply by π.

Then π (R + r) (R − r) = πCP²

But π(R + r) (R − r) = area of the ring (Sect. XX. (h)).

And πCP² = area of circle, radius CP (Sect. XX. (b)).

$\therefore$ Area of the ring = area of circle whose diameter is CD.

XXV.—MISCELLANEOUS EXERCISES.

(*See* Easy Exercises IX.)

I.

1. A gentleman bought a semicircular plot of ground for 500 guineas at £122 10*s.* per acre: how much will the fencing cost at 5*s.* 6*d.* per yard?

2. If the radius of a circle is 25 ft., find the number of square feet in a sector whose arc subtends an angle of 157° 30′ at the centre.

3. The diameter of a circle is 12·191 in.: what is the length of an arc subtending an angle of 47° at the centre?

4. Find the area of a parallelogram ABCD, having given AB = 557 ft., BC = 665 ft., and CE (the perpendicular distance of the diagonal BD from C) = 532 ft.

5. What is the area of an hexagon inscribed in a circle of 9 in. radius?

6. The base of a circular segment is 16 in., and the height is 3 in.: find the diameter of the circle of which the segment is a part.

II.

1. In a triangle ABC, AB = 340 ft., AC = 429 ft., and a perpendicular BD on AC divides AC into segments, of which DC is 21 ft. longer than AD: find BC.

2. How many inches long is an arc of 36° 47′ at the centre of a circle whose radius is 2 ft. 4¼ in.?

3. A circus ring is 30 ft. across, and is carpeted for 22 ft. from the centre, leaving an outer ring of tan: what is the area of the tan?

4. A sector is cut from a circular plate of 3 ft. in diameter; the angle of the sector measures 75°: what is the area of the sector?

5. What is the length of carpet 3 ft. 6 in. wide, which will cover a room 18 ft. by 16 ft., so as to leave a space one foot wide all round the room?

6. What will the painting of the walls of a room 15 ft. 6 in. long, 14 ft. broad, and 12 ft. 6 in. high, cost at 5*d.* per square yard?

III.

1. The area of a quadrilateral field ABCD is 3·12 ac.; the lines AE, CF, perpendiculars to the diagonal BD, are 420 lks. and 180 lks. respectively, and the side AB is 580 lks.: find the lengths of BD and DA.

2. One end of a string 14 ft. long is fastened to a stake, and a circle described with the other end: what will be the area of the circle?

3. The radius of a circle is 2 ft.: what will be the area of a sector having its angle 20°?

4. If the cost of paving a floor $12\frac{1}{2}$ yds. long by 6 yds. $4\frac{1}{2}$ in. wide with tiles 7 in. square be £70 17*s.* 6*d.*, find the price of the tiles per 100.

5. The diameter of a circle is 12 ft.: find the area of a square inscribed in it.

6. Draw the plan and find the area of a field from the following measurements:—

	To F	
	350	
To E 0	300	
To D 120	250	
	225	150 to G
To C 50	125	
	100	120 to H
To B 100	75	
	From A	go west

IV.

1. A circular piece of ground is 16 yds. in radius, and consists of a circular flower-bed surrounded by a gravel walk of uniform width, whose area is equal to half that of the flower-bed: required the width of the walk.

2. If the chord of a segment be 48 ft., and the height of the segment 6 ft., what is the area of the segment?

3. On opposite sides of a base, which is 120 yds. long, two isosceles triangles are drawn; the height of one is double the height of the other, and the triangle that has the least altitude has a right angle for its angle opposite the base: find the area of the quadrilateral thus formed in acres, etc.

4. Find the side of an equilateral triangle which cost as much to pave the area at 1*s.* per square foot as to fence the sides at 6*s.* 6*d.* a foot.

5. The radius of a circle is $\sqrt{2}$ in.; two parallel lines are drawn in it, each an inch from the centre: find the area between these lines.

6. What will it cost to carpet a room 17 ft. 6 in. long by 14 ft. 9 in. broad with carpet 2 ft. wide at 5*s.* 6*d.* a yard?

V.

1. The angle of a sector of a circle is $27\frac{1}{2}°$, and the diameter of the circle is $27\frac{1}{2}$ in.: what is the area of the sector?

2. How many feet in height is an isosceles triangle of which the base is 253 in., and the area equal to that of another triangle whose sides are 184 in., 165 in., and 157 in. respectively?

3. Find the area of a circle traced on a sheet of paper by a pair of compasses whose legs are 6 in. long, and contain an angle of 120°.

4. The diameter of a circular cricket-ground is 624 yds.: what would it cost to make a road round it on the outside 54 ft. wide at 8*d.* per square yard?

5. The perimeter of one square is 748 in., and that of another is 336 in.: find the perimeter of a square equal in area to the two.

6. A rectangle is 8 ft. long and 7 ft. broad: find the area of a circle having the same perimeter.

VI.

1. A circle 71 ft. in circumference is divided into two segments in the ratio of 32 : 81: what are the areas of the segments?

2. A rectangle is three times as long as it is broad, and its area is 60 sq. ft. $27\frac{3}{16}$ sq. in.: find its dimensions.

3. The radius of a circle is 10 ft.; two parallel chords are drawn, each equal to the radius: find the area of the zone between the chords.

4. A plan of an estate is drawn on a scale of 1 in. to 20 ft.: find what space on the map will correspond to 8000 sq. yds. of the estate.

5. A ladder 25 ft. long stands upright against a wall: how far must the foot of the ladder be drawn out so as to lower the top half a foot?

6. A regular polygon of 12 sides is inscribed in a circle of which the radius is 1 ft.: find the area of the polygon.

VII.

1. The area of a regular hexagon circumscribed by a circle is 122·3 sq. in. less than that of the circle: find a side of the hexagon.

2. What length of rope will tether a goat in $\frac{1}{4}$ ac. of grass?

3. The cost of paving an elliptical courtyard was £23 2*s.* at 3*s.* 6*d.* a square yard; the minor axis measured 36 ft.: find the length of the major axis.

4. The sides of a right-angled triangular field, including the right angle, are 357 ft. and 476 ft.: find the area of the field.

5. A circle is 4 ft. in circumference: find the area of the square inscribed in it.

6. Find the side of a regular hexagon which shall be equal in area to an equilateral triangle, each side of which is 150 ft.

VIII.

1. The area of an isosceles triangle is $56\frac{1}{4}$ sq. in.; each angle at the base is 45°: find the length of the base.

2. The superficial ring containing the minute divisions on the face of a clock has for its outer diameter 9·6 in., and for its inner diameter 9·48 in.: what decimal of a square inch is the space occupied in each division?

3. An equilateral triangle and a square have the same areas: compare their perimeters.

4. Find the cost of paving a quadrilateral courtyard whose diagonal is

85 ft., and the perpendiculars on it from the opposite corners 34 ft. and 40 ft., at 6*s.* a square yard.

5. ABC is a quadrant of a circle whose centre is A; the radius AB is $7\frac{1}{2}$ yds.: find the length of the arc.

6. Find the rent of a square field whose diagonal is 7 ch. 15 lks. at £2 5*s.* per acre.

IX.

1. What is the area of a triangle ABC, its sides being AB = 293, AC = 234, BC = 85? How far must the base AC be produced to meet BD, the perpendicular altitude?

2. The side of a square is 16 ft.; a circle is inscribed in the square so as to touch all its sides: find the area between the circle and the square.

3. A circle and a square have the same perimeter: compare their areas.

4. What will it cost to carpet a room 15 ft. 8 in. long, and 13 ft. 6 in. broad, with carpet 3 ft. 6 in. wide at 4*s.* 9*d.* a yard?

5. Find the area of the segment of a circle, the angle of the sector being 30°, and the radius of the circle 10 ft.

6. If the sides of a triangle are 26 in., 28 in., and 30 in., respectively, find the diameter of the circumscribed circle.

X.

1. The angle B in the triangle ABC is a right angle; AB = 25, BC = 15; from a point D in AC, a line DE is drawn perpendicular to AB, making AE = 18: find the area of the triangle AED.

2. Two unequal circles touch one another, the sum of their circumferences being 88 in.: what is the distance between their centres?

3. The chord of a sector of a circle is 198, and the diameter is 280: find the area of the sector.

4. How much ground, in a semicircular garden, is enclosed by 249 yds. of fencing?

5. Two square fields jointly contain 6 ac., and the side of one is three-fourths as long as that of the other: how many acres are there in each field?

6. One of the parallel sides of a trapezoid is 2 in. longer than the other, the perpendicular breadth is 7 in., and the area $66\frac{1}{2}$ sq. in.: find the lengths of the two parallels.

BOOK III.

XXVI.—ON THE MEASUREMENT OF SOLIDS.

A solid is a body having three measurements or dimensions, length, breadth, and thickness.

Table of Solid or Cubic Measure.

$12 \times 12 \times 12 = 12^3 = 1728$ cubic inches = 1 cubic foot.
$3 \times 3 \times 3 = 3^3 = 27$ cubic feet = 1 cubic yard.

(*a*) The product arising from the multiplication of measurements of three dimensions may be found in several ways.

(i.) By FRACTIONS.

Required the product of 3 ft. 9 in. by 2 ft. 8 in. by 4 ft. 6 in.

3 ft. 9 in. × 2 ft. 8 in. × 4 ft. 6 in. $= 3\frac{3}{4} \times 2\frac{2}{3} \times 4\frac{1}{2}$
$= \frac{15}{4} \times \frac{8}{3} \times \frac{9}{2} = 45$ cub. ft. Ans.

(ii.) By REDUCTION.

The same example as before will be
45 in. × 32 in. × 54 in. = 77760 cub. in. = 45 cub. ft. Ans.

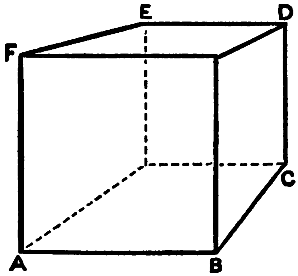
(iii.) By DUODECIMALS.

Table of Duodecimals.

12 cubic inches = $\frac{1}{144}$ of a solid foot = one solid second.
144 cubic inches = $\frac{12}{144}$ or $\frac{1}{12}$ of a solid foot = one solid prime.
1728 cubic inches = one solid or cubic foot.

The unit of volume is a cubic inch, which is a solid, measuring one inch in length, one inch in perpendicular breadth, and one inch in perpendicular height.

Thus ABCDEF is a solid or cubic inch, having AB, BC, and CD each one inch.



12 of these cubes make one solid second.

144 of these cubes, or 12 solid seconds, make a solid prime.

1728 of these cubes, or 12 solid primes, make one cubic foot.

A solid second, therefore, is a solid which measures 12 in. long, 1 in. broad, and 1 in. high.

A solid prime, therefore, is a solid which measures 12 in. long, 12 in. broad, 1 in. high.

And a solid or cubic foot measures 1 ft. long, 1 ft. broad, and 1 ft. high.

Therefore the product of—

Square inches × inches = cubic inches.
Square inches × feet = solid seconds.
Superficial primes × inches = solid seconds.
Square feet × inches = solid primes.
Superficial primes × feet = solid primes.
Square feet × feet = cubic feet.

Example.—Required the product of 3 ft. 9 in. by 4 ft. 6 in. by 5 ft. 4 in.

ft.	in.			
3	9			
4	6			
15	0			= 3 ft. 9 in. × 4 ft.
1	10	6		= 3 ft. 9 in. × 6 in.
16	10	6		= 3 ft. 9 in. × 4 ft. 6 in.
5	4			
84	4	6		= 16 sq. ft. 10′ 6″ × 5 ft.
5	7	6	0	= 16 sq. ft. 10′ 6″ × 4 in.
90	0	0	0	= 16 sq. ft. 10′ 6″ × 5 ft. 4 in. Ans.

(*b*) The quotient arising from the division of a measurement of volume by a measurement of length must give a measurement in area.

Thus cubic inches ÷ inches give square inches.
Cubic feet ÷ feet give square feet.

Again, the quotient arising from the division of a measurement of volume by a measurement of area must give a measurement of length.

Thus cubic inches ÷ square inches give inches.
Cubic feet ÷ square feet give feet.

The results, as in multiplication, are obtained in various ways.

Example.—Required the quotient obtained by dividing 85 cub. ft. 696 cub. in. by 4 ft. 4 in.

(i.) By FRACTIONS.

$85\frac{696}{1728} \div 4\frac{1}{3} = \frac{147576}{1728}$ cub. ft. $\times \frac{3}{13}$ ft. $= \frac{147576}{7488}$ sq. ft. = 19 sq. ft. 102 sq. in. Ans.

(ii.) By REDUCTION.

85 cub. ft. 696 cub. in. ÷ 4 ft. 4 in. = 147576 cub. in. ÷ 52 in. = 2838 sq. in.
= 19 sq. ft. 102 sq. in. Ans.

(*c*) The unit of volume is a cubic inch, and it has been found by actual experiment that such a cube of pure water weighs 252·458 grs.

∴ One cubic foot of pure water weighs 252·458 grs. × 1728.

But an avoirdupois pound weighs 7000 grs.

∴ A cubic foot of pure water weighs $\frac{252{\cdot}458 \times 1728}{7000}$ lbs. avoir.

$$= \frac{252{\cdot}458 \times 1728 \times 16}{7000} \text{ ozs.} = 997{\cdot}13 \text{ ozs.}$$

This is so near 1000 *ozs. that a cubic foot of water is generally considered to weigh* 1000 *ozs.*, or $62\frac{1}{2}$ *lbs.*

Again, by actual experiment, it has been found that a gallon of water weighs 10 lbs. = 7000 grs. × 10.

But one cubic inch of pure water weighs 252·458 grs.

∴ Number of cubic inches occupied by one gallon of water ... ... $= \frac{10 \times 7000}{252{\cdot}458}$

= 277·274 cub. ins.

This result is so near $277\frac{1}{4}$, *that a gallon of water is generally considered to occupy* $277\frac{1}{4}$ *cub. in.*

(*See* EASY EXERCISES X.)

EXERCISE XXI.

1. Find the cube of 2 ft. 8 in.
2. Find the product of $54\frac{1}{2}$ ft. by 5 ft. by 2 ft. 5 in.
3. How many cubic inches are there in $22\frac{1}{2}$ cub. yds.?
4. How many cubic yards are there in $3\frac{1}{2}$ million cubic inches?

5. A vessel holds 15 gallons: what are its cubical contents?

6. A cistern holds 320 gallons of water: find its weight.

7. How many cubic inches of water weigh a ton?

8. What is the weight of 1000 cub. in. of water?

9. What is the volume of 1000 grs. of water?

10. Find the product of 6 yds. 2 ft. 7 in. by 3 ft. 4 in. by 2 ft. 11 in.

11. Find the product of 12 sq. ft. 80 sq. in. by 2 ft. 7 in.

12. Divide 155 cub. ft. by 5 ft. 4 in.

13. Divide 241 cub. ft. 864 cub. in. by 5 ft. 9 in.

14. How many cubic inches are there in a vessel which holds one and a half pints of water?

15. What weight of water will a vessel hold whose volume is $4\frac{1}{4}$ cub. ft.?

16. What quantity of water will weigh 1 ton?

17. A ton of sea-water measures 35 cub. ft.: what is the weight of a gallon of sea-water?

18. Find the volume in cubic inches of a pint of water correct to the fourth place of decimals.

19. What is the weight of a bar of iron 8 ft. by 2 ft. by 3 in., its specific gravity being three times that of water?

20. What quantity of water will a vessel hold whose capacity is $1\frac{1}{2}$ cub. ft.?

21. Multiply 5·01 in. by 3·5 in. by ·025 in.

22. A vessel holds 1000 gallons of water: what is its volume?

23. What is the number of 3-in. cubes, which can be made out of a solid cube whose side is 1 ft.?

24. What must be the side of a cubical cistern to hold 500 gals. of water?

XXVII.—ON THE PARALLELOPIPED.

Definitions.

A *parallelopiped* is a solid bounded by six parallelograms, of which each opposite two are equal and in parallel planes.

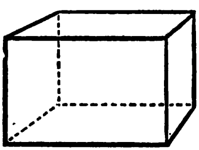
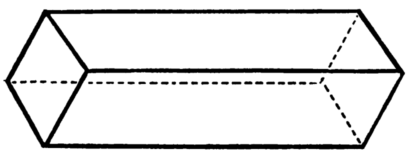


Fig. 1.



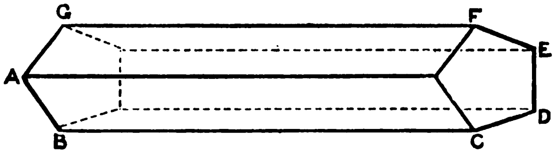
A *prism* is a solid whose sides are parallelograms, but whose ends are any rectilineal figure.

Thus prisms may be triangular, pentagonal, hexagonal, etc.

Thus the figure ABCEG is a pentagonal prism, having its sides parallelograms, but its ends pentagons.

Hence a parallelopiped is one form of prism having its ends also parallelograms.

A *rectangular parallelopiped* is one whose six faces or bounding parallelograms are rectangles. Thus Fig. 1 is a rectangular parallelopiped.

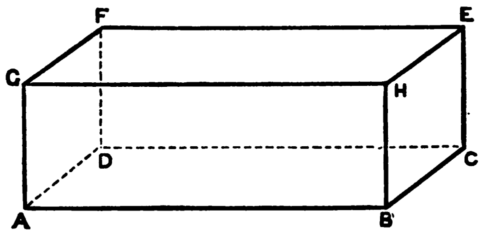


A *cube* is a rectangular parallelopiped whose six faces are squares.

(*a*) To find the surface of a rectangular parallelopiped.

Let ABCEFG be a rectangular parallelopiped.

Let AB = length = a, BC = breadth = b, BH = height = h.



$$\left.\begin{array}{rl} \text{Then area of surface ABCD} &= ab \\ \text{,, \quad ,, \quad GHEF} &= ab \\ \text{,, \quad ,, \quad BHCE} &= bh \\ \text{,, \quad ,, \quad ADFG} &= bh \\ \text{,, \quad ,, \quad ABHG} &= ah \\ \text{,, \quad ,, \quad DCEF} &= ah \end{array}\right\}$$

$$\therefore \text{ Total surface} = 2\,(ab + bh + ah)$$

Rule.—Take the sum of the area of the six faces.

(*b*) To find the surface of a cube.

If the parallelopiped be a cube,

$$\text{Then } AB = BC = BH = a$$
$$\text{Total surface} = 2\,(a^2 + a^2 + a^2) = 6a^2$$

RULE.—**Multiply the square of the length of the side by six.**

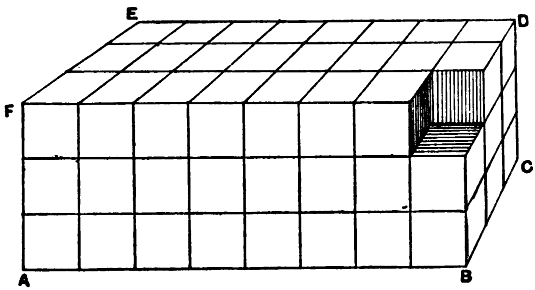
(*c*) To find the volume of a rectangular parallelopiped.

The solid may be considered as composed of thin laminæ of the area of the base placed over one another to the height given.

Let a = length, b = breadth, and h = height.

$$\text{Then area of base} = ab$$
$$\therefore \text{Volume} = ab \times h$$

In the parallelopiped ABCDEF, if the length AB is 8 units, and the



breadth BC 3 units, and the height AF 3 units, it is evident that the volume = 8 × 3 × 3 cubic units = 72 cubic units.

RULE.—**Find the continued product of the length, breadth, and height;** *or*, **Multiply the area of the base by the height.**

(*d*) To find the volume of a cube.

If the parallelopiped be a cube,

$$\text{Then length} = \text{breadth} = \text{height} = a$$
$$\therefore \text{Volume of cube} = a \times a \times a = a^3$$

RULE.—**Take the cube of the length of the side.**

(e) To find the height of a parallelopiped, the area of the base and the volume being given.

Let V = volume, and A = area of the base, and h = height.

$$\text{Now } A \times h = V \qquad \therefore h = \frac{V}{A}$$

RULE.—**Divide the volume by the area of the base.**

(f) To find the area of the base of a parallelopiped, the volume and height being given.

$$\text{Now } A \times h = V \qquad \therefore A = \frac{V}{h}$$

RULE.—**Divide the volume by the height.**

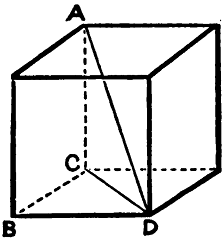
(g) To find the side of a cube, the volume being given.

Let V = volume, and s = length of side.

$$\text{Now } s^3 = V \qquad \therefore s = \sqrt[3]{V}$$

RULE.—**Take the cube root of the volume.**

(h) To find the diagonal of a cube, the side being given.



Let side of the cube BD = s.

Then CD = diagonal of the base = $s\sqrt{2}$

AD = diagonal of the cube

$\therefore AD^2 = AC^2 + CD^2 = s^2 + 2s^2 = 3s^2$

$\therefore AD = s\sqrt{3}$

RULE.—**Multiply the side by the square root of 3.**

(*See* EASY EXERCISES XI. A, B, AND C.

EXERCISE XXII.

1. Find the volume of a rectangular parallelopiped 7 ft. 9 in. long, 4 ft. 6 in. broad, and 2 ft. 3 in. deep.

2. Find the volume of a cube whose side is 15 ft. 4 in.

3. Find the number of cubic feet in a room 11 ft. 8 in. long, 10 ft. 5 in. wide, and 9 ft. 3 in. high.

4. The area of the base of a rectangular parallelopiped is 1000 sq. in.; its height is 1 yd.: find its volume.

5. Find the weight of 8 fir planks, each 6 ft. 6 in. long, 11¼ in. wide., and 4 in. thick, at 33 lbs. the cubic foot.

6. Find the length of the edge of a cube whose volume is 110 cub. in.

7. What weight of water, to the nearest cwt., will a rectangular cistern 5 ft. 6 in. long, 5 ft. 6 in. wide, and 5 ft. 3 in. deep hold?

8. How many gallons of water will a rectangular cistern 9 ft. 9 in. long, 4 ft. 6 in. broad, and 3 ft. 5 in. deep hold?

9. Find the weight of a cubic fathom of water.

10. A cubic foot of wood weighs 20 lbs.: find the weight of 10 planks, each 30 ft. long, 1 ft. wide, and 1 in. thick.

11. What is the length of the edge of a cubical cistern which contains as much as a rectangular one whose edges are 154 ft. 11 in., 70 ft. 7 in., and 53 ft. 1 in.?

12. What is the weight to the nearest pound of the plating of an iron tank open at the top, 27 ft. square at the base, and 9 ft. high? The plating is $\frac{7}{16}$ of an inch thick, and $4\frac{1}{2}$ per cent. is to be allowed for rivets and overlapping of plates. The iron to be reckoned at 480 lbs. the cubic foot.

13. A piece of copper 1 ft. long, 9 in. wide, and $\frac{5}{8}$ in. thick, is rolled into a plate 6 ft. long and 4 ft. wide: how thick will the plate be?

14. A cubic foot of copper weighs 560 lbs. It is rolled into a square bar 40 ft. long. An exact cube is cut from the bar: what is its weight in pounds?

15. Find the diagonal of a cube the edge of which is 12 ft.

16. The diagonal of a cube is 15 ft.: find its edge.

17. Find the total surface of a parallelopiped whose length is 9 ft. 5 in., breadth 3 ft. 5 in., and height 2 ft. 4 in.

18. Find the surface of a cube whose side is 5 ft. 10 in.

19. A cistern is partly filled and has 150 gallons of water in it: if the length of the cistern be 4 ft. and the breadth 3 ft., find the depth of the water.

20. The three edges of a rectangular parallelopiped that meet in an angle are respectively 25 ft., 54 ft., and 160 ft.: find the side of a cube which has the same volume.

21. A cube contains 11 cub. ft. 675 cub. in.: find the length of its edge.

22. The contents of two cubes are respectively 5359·375 cub. ft. and 5·359375 cub. ft.: find the difference of the lengths of their edges in inches.

23. The three edges of a rectangular parallelopiped are 3 ft., 2·52 ft., and 1·523 ft.: find its volume.

24. A cubical block contains 1 cub. yd. 2 cub. ft. 541 cub. in.: find the cost of covering its entire surface with lead at 1*s*. 6*d*. per square foot.

25. A log of timber is 18 ft long, 18 in. broad, and 14 in. thick: if $2\frac{1}{2}$ solid feet be cut off the end of it, what length is left?

26. How many gallons will a cistern hold whose length, breadth, and depth are 4 ft. 9 in., 3 ft. 6 in., and 2 ft. 9 in. respectively?

27. The diagonal of a cube is 5 ft.: find its volume.

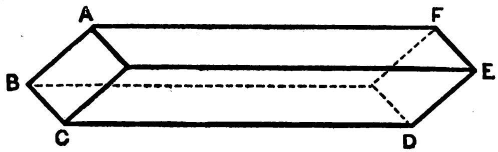
28. What depth must a cistern 7 ft. long by 4 ft. broad be to hold 2500 gals. of water.

29. Find the diagonal of a cube whose volume is 1000 cub. ft.

30. What quantity of water will a cistern 4 ft. long, 3 ft. 6 in. broad, and 3 ft. deep hold?

XXVIII.—ON THE OBLIQUE PARALLELOPIPED.

An *oblique parallelopiped* is one which has its edges oblique to the base; that is, its faces are not rectangles.



Thus ABCDEF is an oblique parallelopiped, and all its angles ABC, AFE, EDC, etc., are either greater or less than right angles.

(*a*) To find the surface of an oblique parallelopiped.

It is evident that the total surface will be as in the case of the rectangular parallelopiped—the sum of the surfaces of its six sides. These sides will be rhomboidal in form instead of rectangular.

RULE.—**Take the sum of the areas of the six faces.**

(*b*) To find the volume of an oblique parallelopiped.

Let ABCDEG be an oblique parallelogram.

It has been shown (Sect. VII.) that an oblique parallelogram is equal in area to a rectangle on the same base and having the same perpendicular altitude, so an oblique parallelopiped has the same volume as the rectangular parallelopiped on the same base and of the same perpendicular height.

Thus the oblique parallelopiped ABCDEG has the same volume as the right parallelopiped FGCDEK, being on equal bases and of the same perpendicular height.

This may also be shown by taking a perpendicular section EKFH and placing the piece EGAFH thus cut off at the other end, DNBMC;

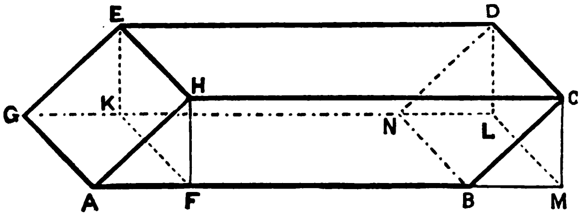


Fig. 2.

thus it is evident that the oblique solid ABCDEG is equal to the right solid FMCDEK.

RULE.—**Multiply the area of the base by the perpendicular height**; *or*, **Multiply the area of the right section by the length of the parallelopiped.**

EXERCISE XXIII.

1. Find the volume of an oblique parallelopiped the area of whose base is a rhombus whose diagonals are 10 ft. and 8 ft., and perpendicular 12 ft.

2. Find the volume of an oblique parallelopiped whose length is 8 ft., and area of a right section is $12\frac{1}{2}$ sq. ft.

3. The area of the base of a parallelopiped is 60 sq. in. its perpendicular height is 3 ft.: find its volume.

4. The dimensions of a right section of an oblique parallelopiped are 8 in. by 6 in., and its length 20 in.: find its volume.

5. In Fig. 2 let AB = 20 in., FH = 4 in., and FK = 6 in.: find the volume.

6. What is the volume of an oblique parallelopiped the area of whose right section is $1\frac{1}{2}$ sq. ft., and whose length is 5 ft.?

7. The right section of a parallelopiped which is 6 ft. long measures 1 ft. by $1\frac{1}{2}$ ft.: find its volume.

8. The area of the base of an oblique parallelopiped is $3\frac{1}{2}$ sq. ft., and its perpendicular height 6 ft.: what is the area of the right section of the solid if its length is 8 ft.?

9. Find the perpendicular height of an oblique parallelopiped whose base is $2\frac{1}{2}$ sq. ft. and length 5 ft., if the area of the right section is 3 sq. ft.

10. Find the volume of an oblique parallelopiped whose base is a rhombus with diagonals 5 in. and 8 in., and whose perpendicular height is 1 ft.

11. Find the volume of an oblique parallelopiped whose length is 10 in., and the area of whose right section is 15 sq. in.

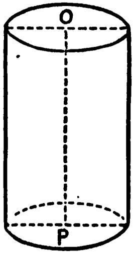
XXIX.—ON RIGHT PRISMS AND CYLINDERS.

A *cylinder* is a solid which is bounded by a circular surface, and whose two ends are parallel.

A *right circular cylinder* is one which has its axis perpendicular to the base, and its base a circle.

Thus Fig. 1 is a right circular cylinder, having the axis OP perpendicular to the base, which is a circle.

Fig. 1.



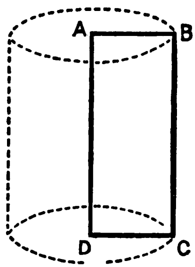


Fig. 2.

The cylinder is produced by the revolution of a rectangle round one of its sides.

Thus in Fig. 2 let ABCD be a rectangle revolving round the side AD, then BC will trace the lateral surface, and AB and DC will trace the circular bases. AD is the *axis* of the cylinder, and DC the *radius.*

A *prism* is a solid whose ends are of equal area and parallel, and whose sides are parallelograms.

A *right prism* has its sides perpendicular to the ends, hence each side is a rectangle.

Thus ABCDEFG in Fig. 3 is a pentagonal right prism, having each of its ends a pentagon and parallel to one another, whilst the five sides are each rectangles.

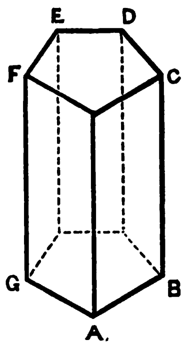


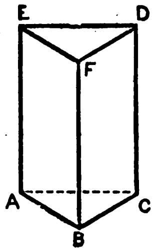
Fig. 3.

(*a*) To find the surface of a right prism.

It is evident that the total surface will be the sum of the area of the two ends and of the sides or faces.

Let ABCDEF be a triangular prism.

Let AB = a, BC = b, AC = c, and the height BF = h.



Then the area of each end $= \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \dfrac{a+b+c}{2}$

Area of face ABFE $= ah$

" " BFCD $= bh$

" " ACDE $= ch$

$\therefore$ Total area $= 2\sqrt{s(s-a)(s-b)(s-c)} + h(a+b+c)$

RULE.—**Take the sum of the areas of the two ends and of the areas of the sides of the prism.**

Fig. 4.

(*b*) To find the volume of a right prism.

The solid may be considered as so many thin laminæ of the area of the base placed one over the other.

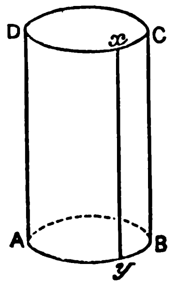
In Fig. 4 let A = area of the base.

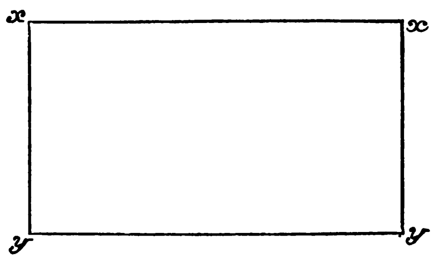
Then volume = A × h

RULE.—**Multiply the area of the base by the height.**

(*c*) To find the lateral surface of a right circular cylinder.

Let ABCD be a right circular cylinder. Suppose it to be hollow and





cut open along the line xy, and the surface then spread out. It is evident it would take the form of the rectangle $xxyy$.

$$\text{Hence lateral surface} = xx \times xy$$
$$\text{But } xx = \text{circumference} = c$$
$$\text{Also } xy = \text{height} = h$$
$$\therefore \text{Lateral surface} = c \times h$$

Let r = radius of the cylinder.

$$\text{Then } c = 2\pi r$$
$$\therefore \text{Lateral surface} = 2\pi rh$$

RULE.—**Multiply the circumference of the cylinder by the height.**

(*d*) To find the total surface of a right circular cylinder.

Total surface = area of ends + area of lateral surface.

$$\text{But area of each end} = \pi r^2$$
$$\text{Also area of lateral surface} = 2\pi rh$$
$$\therefore \text{Total surface} = 2\pi r^2 + 2\pi rh = 2\pi r(h + r)$$
$$\text{But } 2\pi r = c$$
$$\therefore \text{Total surface} = c(h + r)$$

RULE.—**Multiply the sum of the radius and height by the circumference.**

(*e*) To find the volume of a right circular cylinder.

The solid may be regarded as composed of many thin laminæ of the area of the base placed over one another to the height of the cylinder.

$$\therefore \text{Volume of cylinder} = \text{area of base} \times h$$
$$\text{But area of the base} = \pi r^2$$
$$\therefore \text{Volume} = \pi r^2 h$$

RULE.—**Multiply the area of the base by the height of the cylinder.**

(*See* EASY EXERCISES XII.)

EXERCISE XXIV.

1. Find the total surface of a triangular right prism whose sides are 2 ft. 3 in., 1 ft. 5 in., and 1 ft. 5 in., and whose height is $1\frac{1}{2}$ ft.

2. Find the lateral surface of an hexagonal right prism, each edge of the base being 2 ft. 3 in., and the height 5 ft.

3. Find the total surface of a right cylinder whose radius is 10 ft. 6 in., and height 2 ft. 4 in.

4. Find the lateral surface of a right cylinder whose diameter is 8 ft. 3 in., and length 6 ft. 8 in.

5. Find the number of cubic feet in a stone column 12 in. in diameter and 12 ft. high.

6. A well is 35 ft. deep, and has a diameter of 3 ft.: what is its capacity?

7. A brewer's vat, cylindrical in form, holds 1000 gals. of water, and its diameter is 7 ft.: find its depth.

8. Find the surface of a right cylinder whose radius is 4 ft., and height 100 ft.

9. Find the volume of a right cylinder whose length is 8 ft. 10 in., and circumference 4 ft. 6 in.

10. The diameter of a well is 3 ft. 9 in., and its depth 45 ft.: what did the excavation cost at 7*s.* 3*d.* per cubic yard?

11. Find the whole surface of an hexagonal right prism $25\frac{1}{4}$ ft. long, the central diagonal of its base being $2\frac{1}{4}$ ft.

12. Find the volume of a triangular right prism whose length is 5 ft., and the sides of whose base are 6 in., 8 in., and 10 in. respectively.

13. A rod of iron, cylindrical in shape, is 42 ft. long, and its diameter is $1\frac{1}{4}$ ft.: find its volume.

14. What quantity of sheet iron is required to make a right cylinder 2 ft. in diameter and 40 ft. long?

15. How many cubic yards were dug out in making a well whose depth was 15 ft., and diameter 3 ft. 6 in.?

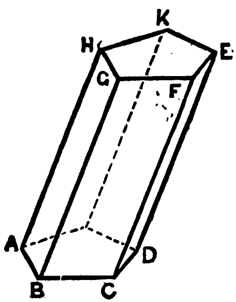
16. A circular shaft is 90 ft. deep and 3 ft. 9 in. in diameter: find the cost of sinking it at 14*s.* 6*d.* a cubic yard.

17. Find the total surface of a triangular right prism whose three sides are 10 in., 6 in., and 8 in., and whose perpendicular height is 15 in.

18. A cylindrical mug is $16\frac{1}{2}$ in. in circumference on the inside, and its depth is 6·4 in.: how many pints will it hold?

19. Find the total surface of a right cylinder whose height is 3 ft., and diameter 10 in.

XXX.—ON OBLIQUE PRISMS AND CYLINDERS.



An *oblique prism* is one whose lateral sides are not at right angles to the base.

Thus ABCDEKH is an oblique prism, having its angles GBC, GBA, etc., each either greater or less than right angles.

An *oblique cylinder* is one whose axis is not perpendicular to the base.

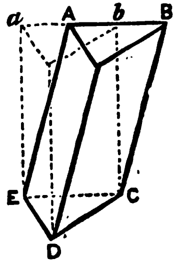
(*a*) To find the surface of an oblique prism.

It is evident that the total surface will be, as in the case of the right prism, the sum of the areas of the two ends and of the area of the sides or faces. These sides or faces, instead of being rectangles as in the rectangular prisms, will be rhomboidal in form.

RULE.—**Add the sum of the areas of the two ends and of the area of the sides of the prism together.**

(*b*) To find the volume of an oblique prism.

It has been shown (Sect. XXVIII.) that an oblique parallelopiped is equal in volume to a right parallelopiped of the same perpendicular height, and by similar reasoning it may be shown that an oblique prism has the same volume as the rectangular prism, having the same base and of the same perpendicular height.

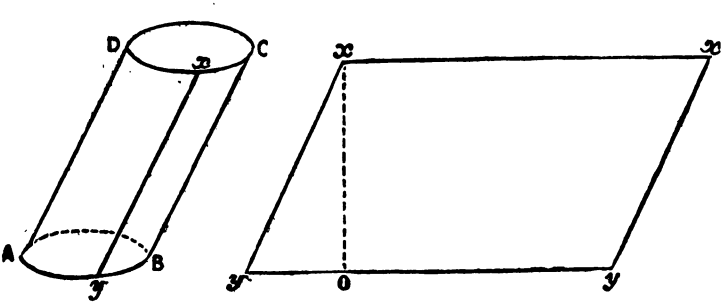


Thus the oblique triangular prism ABCDE is of the same volume as the rectangular prism EDC*ba* upon the same base EDC, and having the same perpendicular height E*a*.

RULE.—**Multiply the area of the base by the perpendicular height.**

(*c*) To find the lateral surface of an oblique cylinder.

Let ABCD be an oblique cylinder. Suppose it to be hollow and cut



open along the line *xy*, and the surface then spread out. It is evident it would take the form of the rhomboid *xxyy*.

Hence lateral surface = $xx \times xo$.

But xx = circumference of cylinder.

Let xo = perpendicular height = h.

$\therefore$ Lateral surface = $c \times h$

Let r = radius of the cylinder.

$$\text{Then } c = 2\pi r$$
$$\therefore \text{ Lateral surface} = 2\pi rh$$

RULE.—**Multiply the circumference of the cylinder by the perpendicular height.**

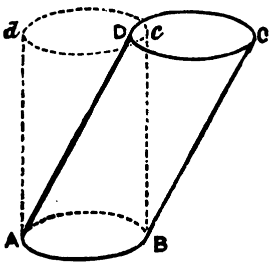
(*d*) To find the total surface of an oblique cylinder.

Total surface = area of two ends + area of lateral surface.

$$\text{But area of each end} = \pi r^2$$
$$\text{Also area of lateral surface} = 2\pi rh$$
$$\therefore \text{ Total surface} = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r(r + h)$$
$$\text{But } 2\pi r = c$$
$$\therefore \text{ Total surface} = c(r + h)$$

RULE.—**Multiply the sum of tho radius and height by the circumference.**

(*e*) To find the volume of an oblique cylinder.



By similar reasoning as in the case of the oblique parallelopiped and prism, it may be shown that an oblique cylinder has the same volume as that of the right cylinder having the same base and the same perpendicular height.

Thus the oblique cylinder ABCD is of the same volume as the right cylinder AB*cd* on the same base and having the same perpendicular height A*d*.

$\therefore$ Volume of cylinder = area of the base × A*d*

Let r = radius of the base, and A*d* = h.

$$\therefore \text{ Volume of cylinder} = \pi r^2 \times h = \pi r^2 h$$

RULE.—**Multiply the area of the base by the perpendicular height.**

NOTE.—*It will thus be seen that the rules for the oblique solids are the same as those for the rectangular solids, if by height is always understood perpendicular height.*

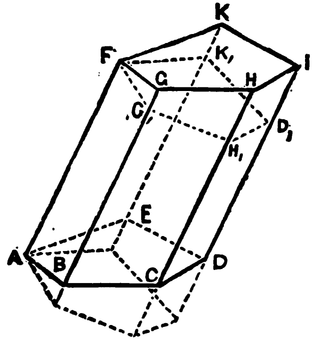
(*f*) To find the volume of any oblique solid, having given the area of a right section and its length.

Take any oblique solid ABCDIKF.

Take a section at right angles to its edges $FG_1H_1D_1K_1$. Suppose this wedge-shaped solid thus cut off $FG_1H_1D_1IK$ to be placed at the other end of the prism F on A, G on B, H on C, etc.

Because the areas of the two ends ABCDE and FGHIK are equal and similar, therefore they will exactly coincide, and the volume of the original oblique solid will equal that of the right solid of the same length.

Thus the volume of the above solid will equal area of $FG_1H_1D_1K_1 \times FA$.



Rule.—**Multiply the area of the right section by the length of the solid.**

EXERCISE XXV.

1. An oblique prism has a base of 23 sq. ft. 115 sq. in., and its perpendicular height is 4 ft. 7 in. : find its volume.

2. Find the volume of an oblique triangular prism whose sides at the base are 7 in., 15 in., and 20 in. respectively, and whose perpendicular height is 45 in.

3. Find the volume of an oblique cylinder, the radius of the base being 2 ft. 6 in., and perpendicular height 4 ft. 3 in.

4. The perpendicular height of an oblique cylinder is 4 ft. 9 in., and the radius of the base 4 ft. 3 in.: find the side of a cube of equal volume.

5. Find the volume of an oblique prism whose base is 3 sq. ft., and perpendicular height 1 ft. 9 in.

6. Find the volume of an oblique cylinder whose radius is 2 ft. 3 in., and perpendicular height 3 ft. 6 in.

7. The circumference of an oblique cylinder at right angles to the axis is 2 ft. 6 in., and its length is 7 ft.: find its lateral surface.

8. An oblique hexagonal prism has the area of its right section 12 sq. ft., and its length is 8 ft.: find its volume.

9. The area of the base of an oblique pentagonal prism is 18 sq. in., and its perpendicular height 10 in.: find its volume.

10. What is the lateral surface of an oblique cylinder whose radius is 1 ft., and perpendicular height 3 ft.?

11. What is the total surface of an oblique cylinder whose diameter is 15 in. at the base, and perpendicular height 2 ft.?

12. Find the volume of an oblique cylinder whose area at the base is $3\frac{1}{4}$ sq. ft., and perpendicular height $2\frac{1}{2}$ ft.

13. Find the lateral surface of an oblique cylinder whose circumference at right angles to the axis is 2 ft., and the length of whose axis is 4 ft.

14. Find the volume of an oblique hexagonal prism whose area at the end is 4 sq. ft., and whose perpendicular height is 6 ft.

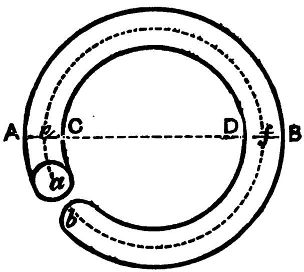
15. What is the area of a right section of an oblique cylinder 5 ft. long, whose area at the base is 4 sq. ft., and perpendicular height 4 ft.?

16. Find the volume of an oblique solid whose perpendicular height is 10 ft., and base a triangle whose sides are 12 in., 6 in., and 10 in. respectively.

XXXI.—ON CIRCULAR RINGS.

(*a*) To find the volume of a circular ring.

The ring may be considered as a cylinder having a circular base, as at *a*. The length or height of the cylinder is the length of the mean circumference of the ring from *a* to *b*.



Let D = diameter of the outer ring AB, and *d* = diameter of the inner ring CD.

Then $\frac{D-d}{2}$ = diameter of the section at *a*

And $\frac{D-d}{4}$ = radius of the section at *a*

$\therefore$ Area of section at $a = \pi\left(\frac{D-d}{4}\right)^2$

Again $\frac{D+d}{2} = ef$ = diameter of mean circumference *aefb*

$\therefore \pi\left(\frac{D+d}{2}\right)$ = length of mean circumference *aefb*

= length or height of supposed cylinder.

$\therefore$ Volume of ring = area of section × length of mean circumference

$$= \pi\left(\frac{D-d}{4}\right)^2 \times \pi\left(\frac{D+d}{2}\right)$$

$$= \frac{\pi^2}{32}(D-d)^2 \times (D+d)$$

$$= \tfrac{1}{32}(\pi D - \pi d)^2 \times (D+d)$$

But πD = circumference of outer circle = C
And πd = circumference of inner circle = c
$\therefore$ Volume of ring $= \frac{1}{32}(C - c)^2 (D + d)$

Rule.—**Multiply the square of the difference of the outer and inner circumferences by the sum of the outer and inner diameters, and take one thirty-second part of the product.**

(b) To find the volume of a flat ring.

If the ring be flat, it may be regarded as the difference between a cylinder whose diameter is AB and another cylinder whose diameter is ab.

Let AP = height of each cylinder = h.

Let AB = D, and $ab = d$.

Then volume of outer cylinder

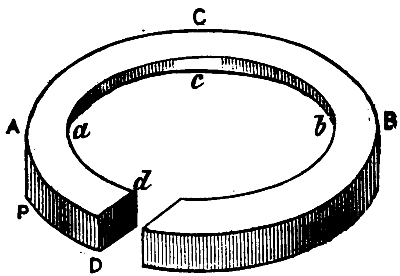
$$ADBC = \pi h \frac{D^2}{4}$$

Then volume of inner cylinder

$$adbc = \pi h \frac{d^2}{4}$$

$\therefore$ Volume of ring

$$= \frac{\pi h}{4}(D^2 - d^2)$$



Rule.—**Multiply the difference of the squares of the outer and inner diameters by one-fourth of the height multiplied by π.**

EXERCISE XXVI.

1. What is the volume of a circular ring whose outer diameter is 2 ft., and inner diameter 1 ft. 8 in.?

2. Find the volume of a circular ring whose outer diameter is 3 in., and thickness of the ring $\frac{1}{5}$ of an inch.

3. Find the volume of a flat ring whose outer diameter is 3 ft., inner diameter 2 ft., and height 3 in.

4. The length of the outer circumference of a gold ring is $1\frac{1}{4}$ in., the circumference of the section of the ring $\frac{1}{3}$ of an inch: find the contents of the ring to the thousandth of a cubic inch ($\pi = 3\frac{1}{7}$).

5. A flat ring has an outer diameter of 4 ft., the thickness of the metal is 3 in., and the height of the ring 2 in.: find its volume.

6. Find the volume of a circular ring whose outer circumference is 3 ft., and inner diameter 8 in.

7. What is the volume of a flat ring whose outer circumference is 3 ft., and inner diameter 8 in., and the height of the ring $1\frac{1}{2}$ in.?

8. A circular ring has an outer circumference of 12 ft.; the thickness of the ring is 3 in.: find its volume.

9. A flat ring has an outer circumference of 12 ft.; the thickness of the ring is 3 in., and it is 3 in. high: find its volume.

10. What is the volume of a circular ring surrounding a circular cistern 5 ft. in circumference, the thickness of the ring being 3 in.?

11. What is the volume of a flat ring surrounding a circular driving-wheel whose circumference is 16 ft., the ring being 6 in. thick and 4 in. deep?

12. A circular cistern 6 ft. in circumference is surrounded by a circular ring 4 in. thick: find the volume of the ring.

13. A flat ring surrounds a wheel 12 ft. in diameter, the ring being 4 in. thick and 6 in. deep: find its volume.

14. A circular ring has an outer circumference of 3 ft., and an inner circumference of $1\frac{1}{2}$ ft.: find its volume.

15. What is the difference in volume between a circular ring and a flat ring, the external and internal diameters of each being 5 ft. and 4 ft., and the height of the flat ring being equal to the thickness of the circular ring?

XXXII.—ON THE PYRAMID.

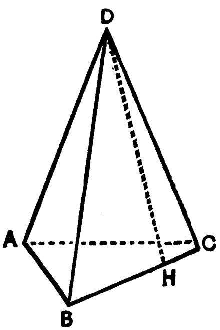
A *pyramid* is a solid whose base is a rectilineal figure, and whose sides are three or more triangles which meet at a point.

Pyramids, therefore, may be triangular, quadrilateral, pentagonal, etc.

(*a*) To find the lateral surface of a pyramid.

Let ABCD be a triangular pyramid.

The area of the lateral surface will equal the area of the sides ABD, BDC, and ADC.



Draw DH perpendicular to BC. This will be the slant height of the pyramid.

$$\text{Area of face ABC} = \frac{BC \times DH}{2}$$

$$\text{Area of face ADB} = \frac{AB \times DH}{2}$$

$$\text{Area of face ADC} = \frac{AC \times DH}{2}$$

$\therefore$ Total lateral surface

$$= (AC + AB + BC) \times \frac{DH}{2}$$

= Perimeter of base × half the slant height.

RULE.—**Multiply the perimeter of the base by one-half the slant height.**

(b) To find the volume of a pyramid.

Let ABCD (Fig. 1) be a pyramid.

Complete the prism BCDFEA, of which the pyramid was a part, by drawing CE and DF parallel to AB, and cutting them by a plane AEF parallel to BCD.

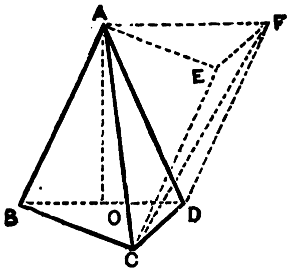


Fig. 1.

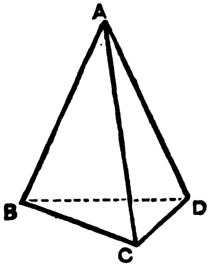


Fig. 2.

Then there are formed three pyramids:

(i.) BCDA on base BCD with apex A (Fig. 2); or on base ABD with apex C.

(ii.) AEFC on base AEF with apex C (Fig. 3).

(iii.) FDCA on base FCD with apex A (Fig. 4); or on base ADF with apex C.

Let AO be the perpendicular height of the prism (Fig. 1).

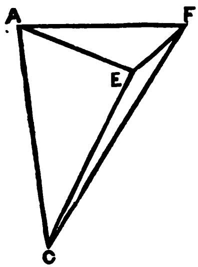


Fig. 3.

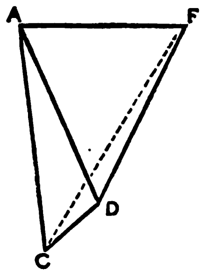


Fig. 4.

Then pyramid BCDA (Fig. 2) = pyramid AEFC (Fig. 3), because they are on equal bases BCD and AEF, and are of the same height AO.

Pyramid BCDA (Fig. 2) = pyramid FCDA (Fig. 4), because they are on equal bases ABD and AFD, and of the same height AO.

Therefore each pyramid is $\frac{1}{3}$ of the prism.
But volume of prism = area of base × height.

$$\therefore \text{Volume of pyramid} = \frac{\text{area of base} \times \text{height}}{3}$$

RULE.—**Multiply the area of the base by one-third of the perpendicular height.**

EXERCISE XXVII.

1. Find the total surface of a square pyramid whose edge measures 3 ft. 6 in., and slant height 5 ft. 4 in.

2. A triangular pyramid on an equilateral base, each of whose sides is 15 ft., has an altitude of 30 ft.: find its volume.

3. The base of a pyramid is a regular hexagon whose side is 16 ft., and whose perpendicular height is 25 ft.: find its volume.

4. A pyramid has a regular hexagon for its base, each side being 20 ft., and its perpendicular height is 12 ft.; find its cubical contents.

5. The great pyramid of Egypt was 481 ft. high, and its base was 764 ft. square: find the volume to the nearest number of cubic yards.

6. It is desired to cover a piece of ground 80 ft. square by a pyramidal tent 30 ft. in perpendicular height. Find the cost of the canvas at $4\frac{1}{2}d.$ per square yard.

7. If a pyramid is cut into two pieces by a plane parallel to the base and half-way between the vertex and the base, compare the volumes of the pieces thus formed.

8. Find the volume of a triangular pyramid, each side of the base being $5\frac{1}{2}$ ft., and the perpendicular height 30 ft.

9. Find the whole surface of a square pyramid, each side of the base being 12 feet, and the slant height 25 ft.

10. Find the solidity of a triangular pyramid whose perpendicular height is 20 yds., and each of whose sides at the base is 6 ft.

11. A triangular pyramid is $19\frac{1}{2}$ ft. high, and the sides of its base are $16\frac{1}{2}$ ft., $18\frac{3}{4}$ ft., and $14\frac{1}{4}$ ft. respectively: find its volume.

12. Find the lateral surface of a pyramid the perimeter of whose base is 15 ft., and slant height 8 ft.

13. Find the lateral surface of a regular hexagonal pyramid whose side at base is 3 ft., and perpendicular height 12 ft.

14. Find the volume of a pyramid whose base is 8 sq. ft., and perpendicular height 10 ft.

15. A pentagonal pyramid is 14 ft. in perimeter, and the perpendicular distance from its apex to the edge of the base is 14 ft.: find its lateral surface.

16. The area of the base of a triangular pyramid is 24 sq. ft., and its perpendicular height is 50 ft.: find its volume.

17. It is required to cover a piece of ground, in the form of a rectangle 15 ft. long by 12 ft. broad, with a pyramidal tent 20 ft. high: how much canvas would be required?

18. Find the cubic contents of a pyramidal tent which covers a rectangular piece of ground 15 ft. by 20 ft., the tent being 25 ft. high.

19. The area of the base of a pyramid 15 ft. high is 5 sq. ft.: compare its volume with that of another pyramid with double the base and half the height.

20. Compare the volumes of a pentagonal pyramid a feet high with that of an hexagonal pyramid b feet high, the area of their bases being equal.

21. If a plane parallel to the base cuts a pyramid one-third of its height from the base, compare the volume of the pieces thus formed.

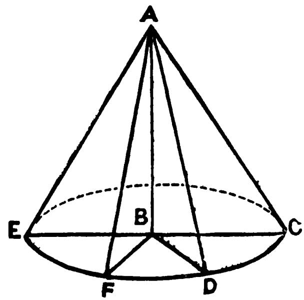
22. If a pyramid 12 in. high be cut by planes parallel to the base at distances of 2 in. and 8 in. from the base, compare the volumes of the pieces thus formed.

23. A pedestal, consisting of a solid block of granite 4 ft. square and 6 ft. high, has a pyramid having an equal base to that of the pedestal and 16 ft. high placed on its top: find the volume of the whole.

24. What will it cost to polish the granite which is exposed to view in the previous question at 2*s.* 6*d.* a square foot?

XXXIII.—ON THE CONE.

A *cone* is a solid formed by the revolution of a right-angled triangle round one of the sides containing the right angle.



Let ABC be a right-angled triangle, having the right angle at B, and let it revolve round AB.

Then AC will trace out a conical surface.

And BC will trace out a circle.

The solid bounded by this conical surface and this circle is called a cone.

A is the *vertex* of the cone.

AC is the *slant height* = AD = AF = AE.

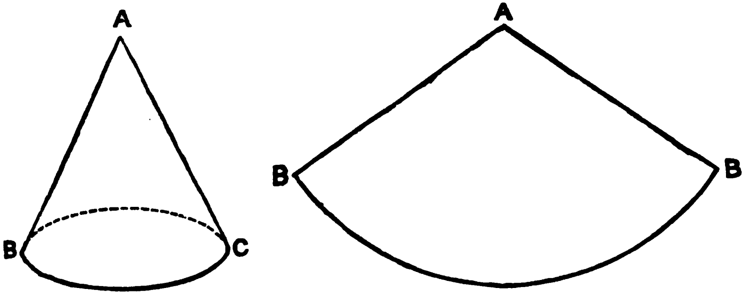
AB is the *height* of the cone.

BC is the *radius* = BD = BF = BE.

(*a*) To find the lateral surface of a cone.

Let ABC be a cone. If it be considered hollow and slit open along the line AB, when spread out it will form a sector of a circle.

The arc of this sector is the circumference of the base of the cone, and the radius of the sector is the slant height of the cone.



$\therefore$ Lateral surface of the cone ABC = area of the sector ABB

$$= \text{arc BB} \times \frac{AB}{2}$$

$$= \text{circumference of cone} \times \frac{\text{slant height}}{2}$$

Let r = radius of cone, and h = slant height.

$$\text{Then lateral surface} = 2\pi r \times \frac{h}{2} = \pi rh$$

RULE.—**Multiply the product of the radius and slant height by π.**

(b) To find the total surface of a cone.

Total surface = area of base + area of lateral surface.

Area of base $= \pi r^2$

Area of Lateral surface $= \pi rh$

$\therefore$ Total surface $= \pi r^2 + \pi rh = \pi r(r + h)$

RULE.—**Multiply the product of the radius and the sum of the radius and slant height by π.**

These last results may also be obtained from the results previously found for the pyramid.

For, as the perimeter of a polygon, inscribed in a circle, becomes the circumference of the circle when the number of the sides of the polygon is increased indefinitely, so the cone may be regarded as a pyramid having for its base a polygon of an indefinite number of sides.

$$\text{Lateral surface of pyramid} = \text{perimeter of base} \times \frac{\text{slant height}}{2}$$

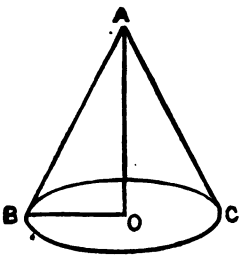
$$\left.\begin{array}{r}\text{But perimeter of base of polygon}\\ \text{of indefinite number of sides}\end{array}\right\} = \text{circumference of base of cone}$$

$$\therefore \text{Lateral surface of cone} = c \times \frac{h}{2} = 2\pi r \times \frac{h}{2} = \pi r h$$

$$\text{And total surface} = \pi r^2 + \pi r h = \pi r(r + h)$$

These are the results previously obtained.

(*c*) To find the slant height, perpendicular height and radius of the cone being given.



Let ABC be the cone. AO = perpendicular height = h.

BO = radius of base = r.

Then AB = slant height.

AOB is a right-angled triangle.

$$\therefore AB^2 = OA^2 + OB^2$$

$$\therefore \text{Slant height} = \sqrt{h^2 + r^2}$$

RULE.—**Take the square root of the sum of the squares of the perpendicular height and the radius of the base.**

(*d*) To find the volume of a cone.

The volume of a cone may be obtained from that of the pyramid. For it may be considered as a pyramid having an indefinite number of sides. Therefore the perimeter of the pyramid ultimately becomes the circumference of the cone.

$\therefore$ Volume of a cone = area of the base × one-third of the perpendicular height.

Let r = radius of base, and h = perpendicular height.

$$\therefore \text{Volume} = \pi r^2 \times \frac{h}{3}$$

RULE.—**Multiply the area of the base by one-third of the perpendicular height.**

(*See* EASY EXERCISES XIII. AND XIV.)

EXERCISE XXVIII.

1. Find the lateral surface of a cone whose diameter is 17 ft. 2 in., and perpendicular height 21 ft.

2. How many yards of canvas $\frac{3}{4}$ yd. wide will be required for a conical tent 20 ft. in diameter and 15 ft. high?

3. Find the volume of a cone the radius of whose base is 4 ft. 6 in., and whose height is equal to its circumference.

4. Find the volume of a cone whose perpendicular height is 7 ft., and the radius of the base 5 ft.

5. What quantity of canvas is necessary for a conical tent whose altitude is 8 ft., and the diameter of whose base is 13 ft.?

6. A cone is 30 ft. high, and the diameter of the base is 1 yd.: find the volume.

7. A cone is 2 ft. high, and its volume is 8 cub. ft.: find the circumference of the base.

8. What is the cost of cleaning the curved surface of a cone whose diameter at the base is $13\frac{1}{3}$ ft., and slant height $18\frac{3}{8}$ ft., at 1*s*. per square foot?

9. Find the volume of a cone whose circumference at the base is 6 yds. 2 ft., and perpendicular height $8\frac{1}{3}$ yds.

10. Find the volume of a cone whose circumference is 11 yds., and slant height $8\frac{3}{4}$ ft.

11. Find the volume of a cone the diameter of whose base is $3\frac{1}{2}$ ft., and whose altitude is 6 ft.

12. The diameter of the base of a cone is 9 ft., and the height is equal to the circumference of the base: find its cost at 1*s*. per cubic foot.

13. The slant height of a cone is 13 ft. 5 in., and the diameter of the base is 7 ft. 4 in.: find the cost of decorating the surface at 1*s*. per square foot.

14. Find the volume of a cone, the circumference of whose base is 12 ft., and whose slant height is 15 ft.

15. What will the canvas cost at 3*s*. 6*d*. a square yard for a conical tent whose height is 10 ft., and circumference at the base 40 ft.?

16. If the radius of the base of a cone whose height is 10 in. is 4 in., find the slant height.

17. What is the total surface of a cone 24 in. in height, and 24 in. in circumference at the base?

18. On a cubical pedestal stands a cone 10 ft. high and 11 ft. in circumference at the base, which touches the edges of the top of the pedestal: find the total contents of the whole.

19. The largest possible cone is cut from a rectangular solid 3 ft. by 2 ft. by 8 ft. high: find the volume of the cone.

20. A cone is cut by a plane parallel to the base and midway between the apex and base: compare the volumes of the solids thus formed.

21. A cone 15 in. high is cut by two planes parallel to the base at distances of 5 in. and 10 in. respectively from the base: compare the volumes of the solids thus formed.

22. What is the volume of the largest possible cone cut out of a cubical block whose edge is 7 ft.?

XXXIV.—ON FRUSTA OF PYRAMIDS.

A *frustum of a pyramid* is the portion of the pyramid included between the base and a plane cutting the solid parallel to the base.

(*a*) To find the lateral surface of a frustum of a pyramid.

Let ABCDE and *abcde* be the two parallel planes of a pyramid PABCD.

Then the area of each face of the frustum is a trapezoid.

Let *pp* be the slant height or perpendicular between the parallel sides AB and *ab*.

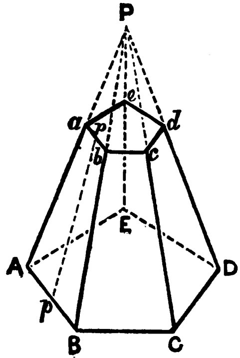
Then area of face AB*ba* $= \frac{(AB + ab) \times pp}{2}$,

and so on for all the faces.

But lateral surface = area of all the faces.

∴ Lateral surface

= (perimeter of base + perimeter of top) $\times \frac{\text{slant height of frustum}}{2}$



RULE.—**Multiply the sum of the perimeters of the base and top by one-half the slant height.**

(*b*) To find the total surface of the frustum of a pyramid.

It is evident that the total surface of the frustum of a pyramid will equal the lateral surface together with the areas of the two ends.

(*c*) To find the volume of the frustum of a pyramid.

Let ABCDEF be a frustum of a pyramid.

It can be cut into three pyramids, ABCE, FEDC, and EFAC.

The volume of the whole frustum will equal the volume of these three pyramids.

Let area of ABC $= S_1$.

" " DEF $= S_2$.

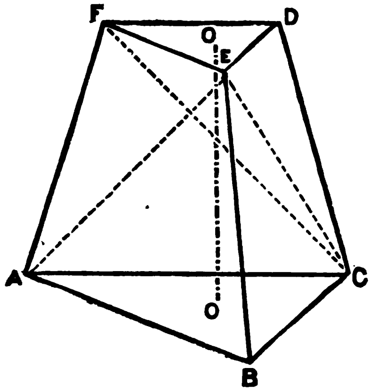
Let height of frustum O*o* $= h$.

Then volume of ABCE $= \frac{S_1 h}{3}$ (Sect. XXXII. (*b*).)

And volume of FEDC $= \frac{S_2 h}{3}$

But it is necessary to get the volume of EFAC in terms of the other two pyramids.

Now, volume of EFAC : volume of FEDC :: Area AFC : Area FDC, or as their bases, being of equal heights O*o*.



Again, area AFC : area of FDC :: AC : FD, or as their sides, being of the same height O*o* (Sect. XII.)

And side AC : FD :: $\sqrt{\text{area of ABC}}$: $\sqrt{\text{area of DEF}}$ :: $\sqrt{S_1} : \sqrt{S_2}$, or as the square root of their areas (Sect. XII.).

∴ Volume of EFAC : volume of FEDC :: $\sqrt{S_1} : \sqrt{S_2}$

∴ Volume of EFAC

$$= \frac{\sqrt{S_1}}{\sqrt{S_2}} \times \text{volume of FEDC}$$

$$= \frac{\sqrt{S_1}}{\sqrt{S^2}} \times \frac{S_2 h}{3}$$

$$= \sqrt{S_1} \times \sqrt{S_2} \times \frac{h}{3} = \frac{h}{3}\sqrt{S_1 S_2}$$

$$\therefore \text{Volume of frustum} = \frac{h}{3}\left\{S_1 + \sqrt{S_1 S_2} + S_2\right\}$$

Rule.—**To the sum of the areas of the two ends add the square root of the product of the areas of the two ends, and then multiply the result by one-third of the perpendicular height.**

Note.—The square root of the product of the areas of the two ends equals the area of the mean section.

(*d*) From the formula $\frac{h}{3}\left\{S_1 + \sqrt{S_1 S_2} + S_2\right\}$—that is, the volume of the frustum—to obtain the formulæ for the volumes of the pyramid and prism.

(i.) If $S_2 = 0$, then the figure becomes a pyramid, and S_2 vanishes.

∴ Volume of pyramid $= \frac{h}{3} S_1$ = area of the base multiplied by one-third of height (Sect. XXXII.)

(ii.) If $S_1 = S_2$, the solid becomes a prism.

$$\therefore \text{ Volume of prism } = \frac{h}{3}(S_1 + \sqrt{S_1 S_1} + S_1) = \frac{h}{3}(S_1 + S_1 + S_1)$$

$$= \frac{h}{3}(3S_1) = hS_1$$

= area of base multiplied by height (Sect. XXVII.).

These results are the same as were obtained in Sections XXXII. and XXVII. respectively.

EXERCISE XXIX.

1. Find the total surface of the frustum of a pyramid whose ends are squares measuring 5 in. and 3 in. in the side, and whose slant height is 12 in.

2. A mound of earth is raised with plane sloping sides; the dimensions at the bottom are 80 yds. by 10 yds., and at the top 70 yds. by 1 yd.; the perpendicular height is 5 yds.: find the cubic contents.

3. A hollow space is excavated in the form of a cistern 6 ft. deep; the area at the top is 100 sq. yds., at the bottom 81 sq. yds.: find the number of gallons it will hold.

4. Find the volume of the frustum of a triangular pyramid, the sides of the base being 9 in., 12 in., and 15 in., and of the top 6 in., 8 in., and 10 in. respectively, and the perpendicular height being 20 in.

5. Find the solid contents of the frustum of a square pyramid, each side of the greater end being 3 ft. 4 in., and of the smaller end 2 ft. 2 in., the perpendicular height being 10 ft.

6. How many cubic feet of water can be contained in a ditch of tde form of an inverted frustum of a pyramid if it measure 400 ft. by 20 ft. at the top, and 300 ft. by 15 ft. at the bottom, the uniform depth being 6 ft.?

7. Find the lateral surface of the frustum of an hexagonal pyramid whose perimeters at top and bottom are 5 ft. and 10 ft. respectively, and whose slant height is $5\frac{1}{2}$ ft.

8. Find the volume of the frustum of a regular hexagonal pyramid whose side at bottom is 3 ft. and at top 2 ft., and whose perpendicular height is 6 ft.

9. Find the total surface of the frustum of a square pyramid whose sides at top and bottom are 3 ft. and 5 ft., and whose slant height is 6 ft. 6 in.

10. A reservoir with slanting sides whose base is 50 ft. by 40 ft., and top 75 ft. by 60 ft., is 15 ft. in perpendicular depth: find the number of gallons of water it will hold.

11. Find the cost of excavating a reservoir whose base measures 20 yds. by 15 yds., and whose top is 30 yds. by 25 yds., and perpendicular depth 6 yds., at 2*s.* 9*d.* a cubic yard.

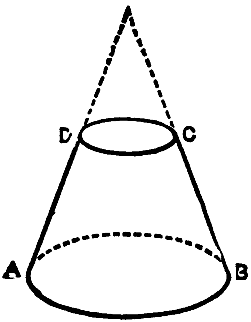
12. Find the lateral surface of the frustum of a square pyramid whose base is 5 ft. square, and top 3 ft. square, and slant height 11 ft.

XXXV.—ON FRUSTA OF CONES.

(*a*) To find the lateral surface of a frustum of a cone.

Let ABCD be the frustum of a cone.

From similar reasoning as before, the circumferences of the top and bottom of the frustum of the cone may be considered as the perimeters of two polygons having an indefinite number of sides.



Let C = circumference and R = radius of the base.

Let c = circumference and r = radius of the top.

Let h = slant height = DA.

Then lateral surface $= (C + c) \times \frac{\text{slant height}}{2}$

(Sect. XXXIV. (*a*).)

But $C = 2\pi R$, and $c = 2\pi r$

$\therefore$ Lateral surface $= 2\pi(R + r) \times \frac{h}{2}$

$= \pi h(R + r)$

Rule.—**Multiply the sum of the radii of the two ends by π times the slant height.**

(*b*) To find the total surface of the frustum of a cone.

The total surface will equal the surface of the bottom + surface at the top + lateral surface.

Area of the base $= \pi R^2$

" " top $= \pi r^2$

Lateral surface $= \pi h(R + r)$

$\therefore$ Total surface $= \pi\{R^2 + r^2 + h(R + r)\}$

Rule.—**To the sum of the squares of the radii of the two ends add the product of the sum of the radii of the two ends by the slant height, and multiply this amount by π.**

(*c*) From the formula $\pi\left\{R^2 + r^2 + h(R + r)\right\}$—that is, the total surface of the frustum of a cone—to obtain the formulæ for the total surface of a cone and cylinder.

(i.) If $r = 0$, then the solid becomes a cone, and r vanishes.

$\therefore$ Total surface of cone $= \pi(R^2 + Rh) = \pi R(R + h)$ (Sect. XXXIII. (*b*).)

(ii.) If $R = r$, then the solid becomes a cylinder.

$$\therefore \text{Total surface of cylinder} = \pi\{R^2 + R^2 + h(2R)\}$$
$$= \pi(2R^2 + 2Rh)$$
$$= 2\pi R(R + h) \quad \text{(Sect. XXIX. (}d\text{).)}$$

These are the same results as were obtained in Sects. XXXIII. and XXIX. respectively.

(*d*) To find the lateral surface of a frustum of a cone, the circumference of the mean section and slant height being given.

From above (*a*) lateral surface of a frustum of a cone $= \frac{C + c}{2} \times$ (slant height).

But $\frac{C + c}{2}$ = circumference of the mean section of the frustum.

$\therefore$ Lateral surface of the frustum = circumference of the mean section × slant height.

Rule.—**Multiply the circumference of the mean section by the slant height.**

(*e*) To find the volume of a frustum of a cone.

Because the circumferences of the top and bottom of the frustum of a cone may be regarded as the perimeters of polygons having an indefinite number of sides, therefore we may, in the formula for the volume of a pyramid, substitute the circumferences of the two ends of the frustum of the cone for the perimeters of the top and bottom of the pyramid respectively.

Let r_1 and r_2 = the radii of the two ends of the frustum.

$$\text{Then } S_1 = \pi r_1^2 \qquad S_2 = \pi r_2^2$$

$$\text{Now volume of frustum of pyramid} = \frac{h}{3}(S_1 + \sqrt{S_1 S_2} + S_2)$$

(Sect. XXXIV. (*c*).)

$$\therefore \text{Volume of frustum of cone} = \frac{h}{3}(\pi r_1^2 + \sqrt{\pi r_1^2 \times \pi r_2^2} + \pi r_2^2)$$
$$= \frac{h}{3}(\pi r_1^2 + \pi\sqrt{r_1^2 r_2^2} + \pi r_2^2)$$
$$= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$$

Rule.—**To the sum of the squares of the radii of the two ends add the product of the radii of the two ends; multiply the result by π times one-third of the perpendicular height.**

(f) From the formula $\frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$—that is, the volume of a frustum of a cone—to obtain the volume of a cone and of a cylinder.

If $r_2 = 0$, then the frustum becomes a cone.

$$\therefore \text{Volume of cone} = \frac{\pi h}{3} r_1^2 = \pi r_1^2 \times \frac{h}{3} \quad \text{(Sect. XXXIII. (}d\text{).)}$$

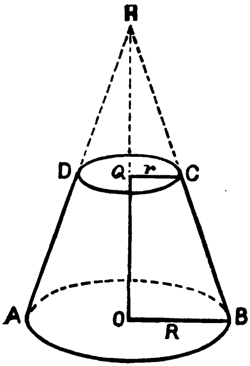
If $r_1 = r_2$, the frustum becomes a cylinder.

$$\therefore \text{Volume of cylinder} = \frac{\pi h}{3}(r_1^2 + r_1 r_1 + r_1^2) = \frac{\pi h}{3}(3r_1^2)$$
$$= \pi h r_1^2 \quad \text{(Sect. XXIX. (}e\text{).}$$

These are the same results as were obtained in the sections on the cone and cylinder respectively.

(g) Another method of finding the volume of a frustum of a pyramid or cone.

The volume of a frustum of a pyramid or cone may be found by taking the difference between the volume of the complete pyramid or cone and that of the part cut off by the plane forming the top of the frustum.



Let ABCD be a frustum of a cone.

Complete the cone HAB.

Then the frustum ABCD may be regarded as the difference between the cone ABH and the cone DCH.

Let the height of the frustum OQ = h.

" radius of the base = R.

" " top = r.

It is necessary to find the height of the cones HAB and HDC.

Let HQ = x.

Then HO = $h + x$

Now HQC and HOB are similar triangles.

$\therefore$ HO : R :: HQ : r

That is $x + h$: R :: x : r

$\therefore Rx = rx + rh$

$$\therefore Rx - rx = rh \qquad \therefore x = \frac{rh}{R - r} = HQ$$

$$\text{But HO} = h + x = h + \frac{rh}{R - r} = \frac{Rh - rh + rh}{R - r} = \frac{Rh}{R - r}$$

$$\therefore \text{Volume of cone HAB} = \tfrac{1}{3}\left(\frac{Rh}{R - r}\right) \times \pi R^2 = \tfrac{1}{3}\left(\frac{\pi R^3 h}{R - r}\right)$$

And volume of cone HDC $= \frac{1}{3}\left(\frac{rh}{R - r}\right) \times \pi r^2$

$$\therefore \text{ Volume of frustum ABCD} = \frac{1}{3}\left(\frac{\pi R^3 h}{R - r}\right) - \frac{1}{3}\left(\frac{\pi r^3 h}{R - r}\right)$$

$$= \frac{\pi h}{3} \times \frac{R^3 - r^3}{R - r} = \frac{\pi h}{3}(R^2 + Rr + r^2)$$

This is the same result as was obtained above (Sect. XXXV.).

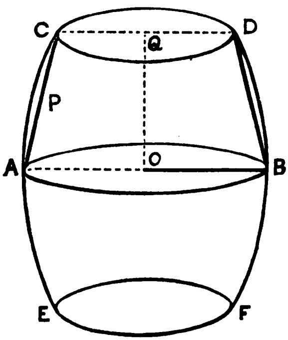
By writing $\pi R^2 = S_2$, and $\pi r^2 = S_1$, we obtain the volume of frustum of pyramid.

$\therefore$ Volume of frustum of pyramid $= \frac{h}{3}(S_2 + \sqrt{S_2 S_1} + S_1)$, which is the same result as was obtained in Sect. XXXIV. (*c*).

(*h*) To find the volume of a cask, or to gauge a cask.

Let EFDC be a cask.

It may be regarded as the frusta of two cones, ABCD and AEFB.



Let R = radius of the bung section OB.

Let r = radius of the end section QD.

Let h = height of the frustum OQ.

Then volume of the frustum ABCD

$$= \frac{h}{3}(\pi R^2 + \pi Rr + \pi r^2)$$

This omits the small volume between the sides of the cask and the section APC.

$\therefore$ Volume of the cask

$$= \frac{2h}{3}(\pi R^2 + \pi Rr + \pi r^2)$$

which gives the volume a little too small.

If for Rr we write R^2,

$$\text{Then volume of the cask} = \frac{2h}{3}(2\pi R^2 + \pi r^2)$$

which gives the volume a little too large.

The sum of these two results divided by 2 will give a result sufficiently correct for all practical purposes.

$$\therefore \text{ Volume of cask} = \frac{1}{2}\left\{\frac{2h}{3}(\pi R^2 + \pi Rr + \pi r^2 + 2\pi R^2 + \pi r^2)\right\}$$

$$= \frac{h}{3}(3\pi R^2 + \pi Rr + 2\pi r^2)$$

$$= \frac{\pi h}{3}(3R^2 + Rr + 2r^2)$$

If instead of h = OQ we take h_1 = height of the cask = 2OQ = $2h$

$$\text{Then volume of cask} = \frac{\pi h_1}{6}(3R^2 + Rr + 2r^2)$$

RULE.—**Find the total sum of three times the square of the radius at the bung, twice the square of the radius at the end, and the product of the bung radius by the end radius. Multiply this sum by π times one-sixth the height of the cask.**

EXERCISE XXX.

1. Find the lateral surface of the frustum of a cone whose top and bottom diameters are 15 ft. 4 in. and 26 ft. 8 in., and whose perpendicular height is 23 ft. 9 in.

2. Find the volume of the frustum of a cone the circumferences of whose bases are 66 ft. and 56 ft., and whose height is 4 ft.

3. Find the weight of water in tons in a tank in the form of a frustum of a cone, the radii of the ends being 10 ft. and 8 ft. respectively, and the perpendicular height being 5 ft.

4. Find the number of cubic feet in the trunk of a tree in the form of a conical frustum whose length is 70 ft., and the diameters of whose ends are 10 ft. and 7 ft.

5. Find the total area of the surface of the frustum of a cone, the diameters of the ends being $2\frac{1}{2}$ ft. and $1\frac{1}{4}$ ft., and the perpendicular height 5 ft.

6. The diameters of the top and bottom of the frustum of a cone are 18 in. and 27 in. respectively, and the perpendicular height is 30 in.: find its volume.

7. If from a right cone, whose slant height is 30 ft. and whose circumference at the base is 10 ft., there be cut off by a plane parallel to the base a cone of 12 ft. in slant height, what is the volume of the frustum?

8. Find the lateral surface of the frustum of a cone the perpendicular height of which is 7 ft., and the radii of the two ends 4 ft. and 5 ft. respectively.

9. Find the volume of a conic frustum the circumferences of whose ends are 66 ft. and 56 ft. respectively, and whose perpendicular height is 6ft.

10. Find the solid contents of a frustum of a cone whose perpendicular height is 7 ft., and the radii of the two ends 4 ft. and 5 ft. respectively.

11. What number of gallons will a cask hold which is 3 ft. high, and whose circumference at the bung is 8 ft. and at the ends 6 ft.?

12. Find the weight of water which a water-cask 6 ft. high, and having its circumference at the centre 12 ft. and at the ends 8 ft., would hold.

13. What is the lateral surface of a frustum of a cone whose circumference of the mean section is 10 in. and slant height $8\frac{1}{2}$ in.?

14. The circumferences of the ends of a frustum of a cone are 2 ft. 6 in. and 6 ft. respectively, and the slant height of the frustum is 4 ft.: find the total surface of the frustum.

15. Find the volume of a cask which is 4 ft. high, and 8 ft. in circumference at the bung and 6 ft. in circumference at the end.

16. If from a cone, whose slant height is 15 ft. and circumference at the base 8 ft, there be cut off by a plane parallel to the base a cone of 5 ft. in slant height, find the volume of the frustum.

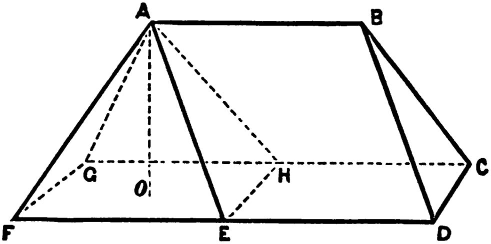
17. How many gallons of water will a cask hold which is 6 ft. high, and whose circumference at each end is 8 ft. and in the middle 10 ft.?

XXXVI.—ON THE WEDGE AND PRISMOID.

(*a*) To find the volume of a wedge.

Let ABCDEF be a wedge having a rectangular base FDCG.

The wedge can be divided into two parts by a plane through A,



parallel to the end BDC, thus forming the pyramid AFEH and the half-prism AEDCB.

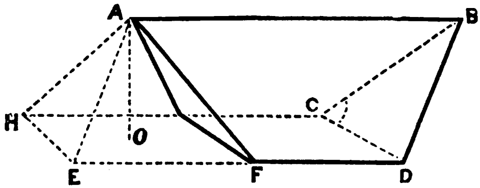
Let AB = the edge = e.
„ FD = length of base = b.
„ DC = breadth of base = d.
„ Ao = height of wedge = h.

Then volume of AFEH $= d(b-e) \times \frac{h}{3}$ (Sect. XXXII. (b).)

„ „ AEDCB $= \frac{deh}{2}$ (Sect. XXVII. (e).)

$$\therefore \text{Volume of wedge} = \frac{deh}{2} + \frac{dh}{3}(b-e)$$
$$= \tfrac{1}{6}\{3deh + 2dh(b-e)\}$$
$$= \tfrac{1}{6}(3deh + 2dbh - 2deh)$$
$$= \tfrac{1}{6}(deh + 2dbh) = \frac{dh}{6}(e + 2b)$$

If the edge of the wedge AB be greater than the base FD, then by completing the half-prism EDBAH it is evident that the volume of the wedge FDBA is equal to the difference between the volumes of the half-prism EDBAH and the pyramid AHEF.



As before, let AB = the edge = e.
" " FD = length of base = b.
" " DC = breadth of base = d.
" " AO = height of wedge = h.

Then volume of pyramid AHEF $= d(e - b) \times \frac{h}{3}$

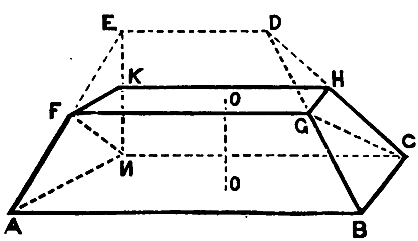
" " prism EDBAH $= \frac{deh}{2}$

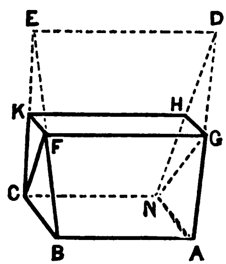
$$\therefore \text{Volume of wedge} = \frac{deh}{2} - \frac{dh(e-b)}{3}$$
$$= \tfrac{1}{6}(3deh - 2deh + 2dbh)$$
$$= \frac{dh}{6}(e + 2b)$$

RULE.—**To the length of the edge add twice the length of the base; multiply the sum by one sixth of the product of the breadth of the base and the height of the wedge.**

(b) To find the volume of a prismoid.

A prismoid is a frustum of a wedge made by a plane passing through the wedge parallel to the base.





Let ABCDE be a wedge, of which ABCN is the base. Let a plane FGHK be passed parallel to the base; then the solid ABHK is a prismoid.

This prismoid can be cut into two wedges by a plane passing through the edges FG and NC, making—

(i.) A wedge whose base is ANCB, and edge FG.

(ii.) A wedge whose base is FGHK, and edge NC.

The volume of the prismoid will evidently be the sum of the volumes of these two wedges.

Let AB = length of base = b.
,, BC = breadth of base = d.
,, FG = length of top = m.
,, FK = breadth of top = n.
,, OO = height of prismoid = h.

Then volume of wedge ABCGF $= \frac{dh}{6}(m + 2b) = \frac{h}{6}(dm + db + db)$

$$= \frac{h}{6}\{(m + b)d + db)\}$$

And volume of wedge HKFNC $= \frac{nh}{6}(b + 2m) = \frac{h}{6}(nb + mn + mn)$

$$= \frac{h}{6}\{(m + b)n + mn\}$$

$$\therefore \text{Volume of prismoid} = \frac{h}{6}\{(m + b)\,d + (m + b)n + db + mn\}$$

$$= \frac{h}{6}\{(m + b)(d + n) + db + mn\} \quad \text{(i.)}$$

But $\frac{m + b}{2}$ = length of mean section of prismoid.

And $\frac{d + n}{2}$ = breadth of mean section of prismoid.

$\therefore$ Area of mean section of prismoid $= \left(\frac{m + b}{2}\right)\left(\frac{d + n}{2}\right) = \text{A}$

$\therefore$ 4 times area of mean section $= 4\text{A} = (m + b)(d + n)$

Substitute this value of $(m + b)(d + n)$ in (i.).

Then volume of prismoid $= \frac{h}{6}(4\text{A} + db + mn)$

RULE.—**To the sum of the areas of the base and top add four times the area of the mean section parallel to these ends, and multiply the sum by one-sixth of the height of the prismoid.**

NOTE.—Haystacks, stone-heaps, railway embankments frequently take this form, and their contents consequently are found by this rule.

EXERCISE XXXI.

1. Find the volume of a wedge the edge of which is $3\frac{1}{2}$ in., the length and breadth of the base 5 in. and 2 in. respectively, and the height 10 in.

2. The length and breadth of the base of a wedge are 10 in. and 5 in., the height is 1 ft., and the edge 1 ft.: find the volume.

3. Find the volume of a wedge the edge of which is 5 in., the length and breadth of the base 8 in. by 3 in., and the height of the wedge 8 in.

4. The area of the base of a wedge is 40 sq. in., the height of the wedge is 6 in., its breadth 4 in., and the length of the edge 8 in.: find the volume of the wedge.

5. Find the total surface of a wedge which has a rectangular base, whose edge is 6 in., slant height 4 in., and length and breadth of base 8 in. by 3 in.

6. Find the side of a cube equal in area to a wedge whose edge is 10 in., length and breadth of base 12 in. and 5 in. respectively, and perpendicular height 6 in.

7. Find the volume of a wedge whose height is 8 in., length and breadth of base 3 in. and 2 in. respectively, and whose edge is 5 in.

8. Find the volume of a vessel in the form of a prismoid, the top being 12 in. by 6 in., the bottom 8 in. by 4 in., and whose depth is 8 in.

9. How many cubic yards are there in a haystack in the form of a prismoid, the dimensions at the base being 40 ft. by 25 ft., at the top 30 ft. by 20 ft., and the height of the stack 40 ft.?

10. A heap of stones in the form of a prismoid measures at the base 15 ft. by 12 ft., and at the top 12 ft. by 10 ft.; its height is 3 ft.: find the number of cubic yards in the heap.

11. A railway embankment is 825 ft. long, and has a uniform width at the top of 32 ft.; at one end it is 19 ft. high, and gradually decreases to the other end to 8 ft. high; the widths at the base at these ends are 108 ft. and 64 ft. respectively: find the number of cubic yards in the embankment.

12. A railway embankment is half a mile long, and has a uniform width of 30 ft. at the top; at one end it is 25 ft. high, and gradually decreases to the other end to 12 ft. high; the widths at the base at the ends are 120 ft. and 80 ft. respectively: find the cost of making the embankment at 8*d*. a cubic yard.

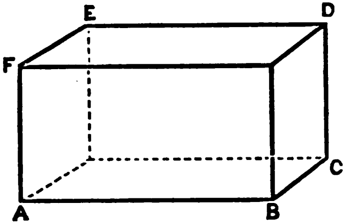
13. What would be the volume of the largest possible wedge cut from a cube whose side is 10. in.?

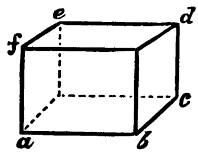
XXXVII.—ON SIMILAR RECTILINEAL SOLIDS.

Similar solids are such as are of the same form, and whose corresponding dimensions are proportional.

(*a*) Rectangular solids are to one another as the product of their dimensions.

Let ABCDEF and *abcdef* be two rectangular solids.





Then volume ABCDEF : volume *abcdef* : : AB × BC × CD : *ab* × *bc* × *cd*.

∴ (i.) *If the bases of two rectangular solids are equal, their volumes are as their heights.*

(ii.) *If the heights of two rectangular solids are equal, their volumes are as their bases.*

Example.—If two cubes have their contents the one double of the other, and the edge of the larger cube is 1 ft.: find the edge of the smaller.

Volume of larger cube : volume of smaller cube : : $12^3 : x^3$

$$2 : 1 :: 12^3 : x^3$$

$$\therefore 2x^3 = 12^3$$

$$\therefore x^3 = \frac{12^3}{2} = \frac{1728}{2} = 864$$

$$\therefore x = \sqrt[3]{864} = \sqrt[3]{8 \times 27 \times 4} = 6\sqrt[3]{4} = 9{\cdot}05 \text{ in.} \quad \text{Ans.}$$

(*b*) The volumes of similar solids are to one another as the cubes of corresponding sides.

Let ABCD and *abcd* be two similar solids.

Then their sides are proportional.

$$\therefore \text{AB} : ab :: \text{BC} : bc :: \text{DO} : do$$

Let AB = *abx*. Then BC = *bcx*; and DO = *dox*.

$$\text{Then volume of ABCD} = \frac{\text{AB} \times \text{BC} \times \text{DO}}{3} = \frac{abx \times bcx \times dox}{3}$$

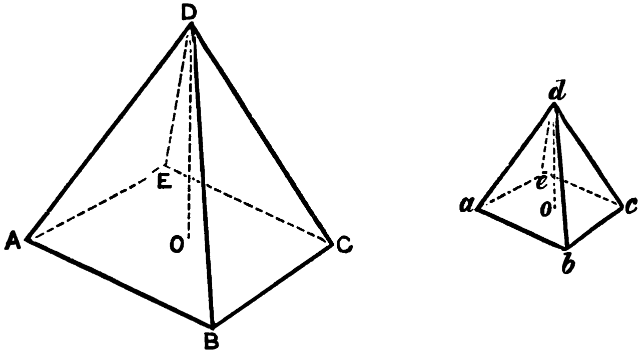
$$= x^3\left(\frac{ab \times bc \times do}{3}\right)$$

And volume of $abcd = \dfrac{ab \times bc \times do}{3}$

$\therefore$ Volume of ABCD : volume $abcd$ :: $x^3 : 1$

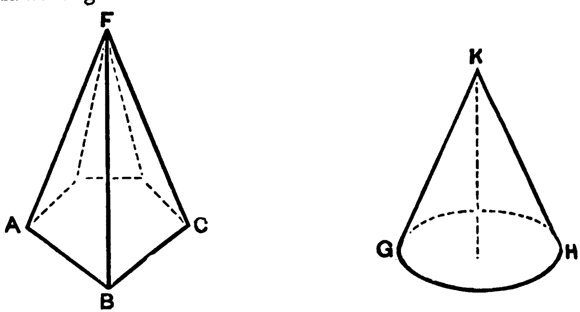
But $x = \dfrac{\text{A}}{ab}$ $\qquad \therefore x^3 : 1 :: \text{AB}^3 : ab^3$

$\therefore$ Volume of ABCD : volume $abcd$ :: $\text{AB}^3 : ab^3$



(c) **Pyramids and cones on equal bases and having equal heights are equal.**

Let the pyramid ABCF and the cone GKH have equal bases and be of the same height.



Let area of bases = A, and height = h.

Area of pyramid = $\frac{1}{3}\text{A}h$

Area of cone = $\frac{1}{3}\text{A}h$

$\therefore$ Volume of pyramid = volume of the cone having an equal base and of the same height.

(*d*) Volumes of cylinders are as the products of the squares of their radii and their heights.

Let V, r, and h = volume, radius, and height of first cylinder.
And V_1, r_1, and h_1 = volume, radius, and height of second cylinder.

$$\text{Then } V = \pi r^2 h \text{ and } V_1 = \pi r_1^2 h_1$$

$$\therefore V : V_1 :: \pi r^2 h : \pi r_1^2 h_1 :: r^2 h : r_1^2 h_1$$

∴ Volumes of cylinders are as the products of the squares of their radii and their heights.

(*e*) Volumes of similar cylinders are as the cubes of their radii, or as the cubes of their heights.

Let V, r, and h = volume, radius, and height of first cylinder.
And V_1, r_1, and h_1 = volume, radius, and height of second cylinder.
Because the cylinders are similar,

$$\therefore r : r_1 :: h : h_1$$

But $V = \pi r^2 h$ and $V_1 = \pi r_1^2 h_1$

$$\therefore V : V_1 :: r^2 h : r_1^2 h_1$$

$$\therefore V : V_1 :: r^3 : r_1^3, \text{ because } r : r_1 :: h : h_1$$

$$\text{And } V : V_1 :: h^3 : h_1^3, \text{ because } r : r_1 :: h : h_1$$

∴ Volumes of similar cylinders are as the cubes of their radii or as the cubes of their heights.

Example.—If it costs 15*s.* to gild a solid whose weight is 14 lbs., what will it cost to gild a similar solid of the same material whose weight is 24 lbs.?

Solids being similar, therefore the volumes are as their weights.

$$\therefore V : V_1 :: 14 : 24 \text{ or } 7 : 12$$

Altitudes of similar solids are as the cube roots of their volumes (Sect. XXXVI. (*b*)).

$$\therefore H : H_1 :: \sqrt[3]{7} : \sqrt[3]{12}.$$

Surfaces of similar solids are as squares of their altitudes or of like sides.

Hence $S : S_1 :: (\sqrt[3]{7})^2 : (\sqrt[3]{12})^2$
But cost of gilding is as the surfaces
∴ Cost of S : cost of $S_1 :: (\sqrt[3]{7})^2 : (\sqrt[3]{12})^2$
That is 15*s.* : cost of $S_1 :: (\sqrt[3]{7})^2 : (\sqrt[3]{2})^2$

$$\therefore \text{Cost of } S_1 = \frac{15 \times (\sqrt[3]{12})^2}{(\sqrt[3]{7})^2} = \frac{15 \times \sqrt[3]{144}}{\sqrt[3]{49}} = 21s.\ 6\tfrac{1}{2}d. \quad \text{Ans.}$$

EXERCISE XXXII.

1. The edge of a cube is 1 ft.: find the number of feet in the edge of another cube of double the volume.

2. The height of a pyramid is 12 ft.; it is required to cut off a frustum which shall be one-fourth of the pyramid: find the height of the frustum.

3. The height of a right circular cylinder is 4 ft.: find the height of a similar cylinder nine times the volume.

4. The breadth of a rectangular prism is 15: what is the breadth of a similar prism twice as large?

5. If one edge of a prism is 5 in., and its volume is 52 cub. in., what is the edge of another similar prism whose volume is 27 cub. in.?

6. If a pyramid 14 ft. high were divided into two equal parts by a plane parallel to the base, what would be the height of the upper portion?

7. A pyramid is cut into two pieces by a plane parallel to the base, midway between the vertex and the base: compare the sizes of the two pieces.

8. The altitude of a cylinder is 20 in., and its diameter is 10 in.: what is the altitude of another cylinder whose solidity is twice as much, and whose diameter is 30 in.?

9. The base of a pyramid is $7\frac{1}{2}$ in. square: required the base of a similar pyramid whose volume is to that of the former as 111 : 11.

10. Compare the volumes of two similar prisms, the perimeters of whose bases are as 100 : 125.

11. There are two similar pyramids whose volumes are 162 cub. in. and 384 cub. in. respectively: if the altitude of the smaller is $4\frac{1}{2}$ in., what is the altitude of the greater?

12. Compare the heights of two rectangular solids whose bases are equal, and whose volumes are 10 cub. ft. and 15 cub. ft. respectively.

13. If two cubes have the contents of the one three times that of the other, and the edge of the smaller is 10 in., find the length of the edge of the larger.

14. If a solid weighing 14 lbs. cost 7*s.* $7\frac{1}{2}$*d.* to gild, what will a similar solid weighing $11\frac{3}{7}$ lbs. cost?

15. What is the ratio of the volumes of two cylinders whose lateral surfaces are equal, and heights 10 ft. and 8 ft. respectively?

16. What is the ratio of the lateral surfaces of two cylinders whose volumes are equal, and radii 3 ft. and 4 ft. respectively?

17. The breadth of a rectangular solid is 12 ft.: what must be the breadth of a similar solid whose volume is three times as great?

18. The area of the base of a prismoid is 46 sq. ft., and its volume 87·9 cub. ft.: find the solidity of a similar prismoid whose base is 32 sq. ft.?

19. Compare the volumes of two similar cones whose circumferences are respectively 15 ft. and 12 ft.

20. Compare the height of a cylinder whose radius is 5 ft. with that of an hexagonal prism whose side is 2 ft., the volumes of both being the same.

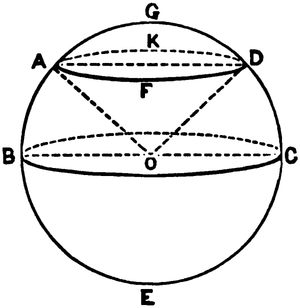
21. A cone is cut into two pieces by a plane parallel to the base midway between the vertex and the base: compare the volumes of the two pieces.

XXXVIII.—ON THE SPHERE.

Definitions.

A *sphere* is a solid bounded by a surface every point of which is equally distant from a point within it called the centre.

A *zone* of a sphere is the portion of a sphere intercepted between two parallel planes. Thus the solid ABCD is a spherical zone.



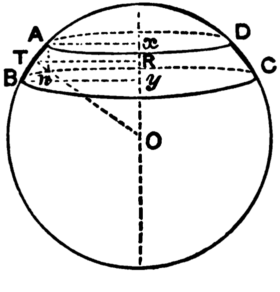
A *segment* of a sphere is the portion of a sphere cut off by a plane passing through it. Thus the plane AFDK cuts the sphere into two spherical segments AFDG, AKDE.

A *sector* of a sphere consists of a spherical segment together with the cone on the plane as base, and whose apex is at the centre of the sphere. Thus AODG is a spherical sector, consisting of the spherical segment AFDG and the cone AKDO.

(a) To find the surface of a sphere, radius given.

Suppose ABCD to be a small zone of the sphere.

It may be regarded as the frustum of a cone whose slant height is AB, if the arc AB be taken very small.



From Section XXXV.(d), the lateral surface of the frustum of a cone = circumference of its middle section × slant height.

∴ Lateral surface of ABCD

= circumference at T × AB

But TR = radius of the section of the sphere at T.

∴ Lateral surface of ABCD

$= 2\pi \text{TR} \times \text{AB}$

Draw An parallel to xy.

Then BAn and TOR are similar triangles having AnB = TRO = each a right angle.

And ATO = a right angle = ATR + RTO = ROT + RTO.

$\therefore$ ATR = ROT, and ATR = ABn

$\therefore$ ROT = ABn, and BAn = RTO

Hence BAn and TOR are similar triangles.

$\therefore$ TR : TO :: An : AB

But An = xy = h = the height of the frustum of the cone.

$\therefore$ TR : TO :: h : AB

If the arc AB is taken indefinitely small, TO = r = radius of the sphere.

$\therefore$ TR : r :: h : AB

$\therefore$ TR $\times$ AB = rh

But lateral surface of ABCD = 2πTR $\times$ AB

Substitute rh = TR $\times$ AB

$\therefore$ Lateral surface of ABCD = $2\pi rh$

If the number of arcs be increased so as to occupy the entire semicircle, then $h = 2r$.

$\therefore$ Sum of all these zones = surface of the sphere $= 2\pi r \times 2r = 4\pi r^2$

RULE.—**Multiply four times the square of the radius by π.**

(b) To find the surface of a sphere, circumference and diameter given.

Surface of a sphere $= 2\pi r \times 2r$

But $2\pi r = c$, and $2r = d$

$\therefore$ Surface of a sphere $= c \times d$

RULE.—**Multiply the circumference by the diameter.**

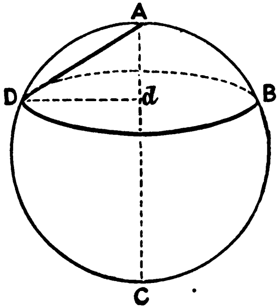
(c) To find the surface of a zone of a sphere, circumference of sphere and height of zone being given.

Area of zone $= 2\pi rh$

But $2\pi r = c$

$\therefore$ Area of zone $= c \times h$

RULE.—**Multiply the circumference of the circle by the height of the zone.**



(d) To find the surface of a spherical segment, the chord of half the arc of the spherical segment being given.

Let AD = chord of half the arc of the spherical segment ADB.

Then surface of spherical segment $= 2\pi r$Ad

But $2r$ = AC

$\therefore$ Surface of spherical segment $= \pi$(AC $\times$ Ad)

But AC $\times$ Ad = AD^2 (Euc. II.)

$\therefore$ Surface of spherical segment $= \pi$AD^2

RULE.—**Multiply the square of half the chord of the arc of the zone by π.**

NOTE.—The chord of half the arc is the radius of the arc which generates the segment of the zone.

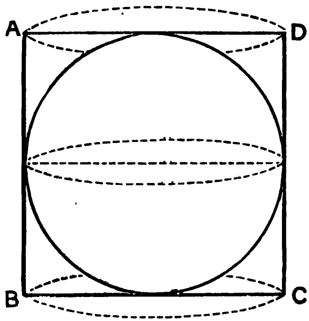
(*e*) Areas of zones of spheres and of segments of spheres are as their heights.

Area of segment ABD = $2\pi r \times Ad$

" " BCD = $2\pi r \times Cd$

∴ Area of segment ABD : area of segment BCD :: $Ad : Cd$

(*f*) To find the surface of a sphere from that of the circumscribing cylinder.



Lateral surface of circumscribing cylinder = $2\pi r \times 2r = 4\pi r^2$

Area of two ends = $2\pi r^2$

∴ Total area of circumscribing cylinder = $6\pi r^2$

But area of the sphere = $4\pi r^2$

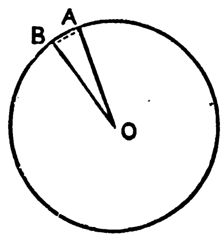
∴ Area of sphere : total area of circumscribing cylinder :: $4\pi r^2 : 6\pi r^2$

∴ Area of sphere = $\frac{2}{3}$ area of circumscribing cylinder

RULE.—**Take two-thirds of the total area of the circumscribing cylinder.**

(*g*) To find the volume of a sphere, radius given.

The sphere may be considered to be composed of a number of small cones, having the diameter of their bases = AB, and height = r = radius of the sphere.



Now volume of cone ABO = area of the base $\times \frac{r}{3}$ (Sect. XXXIII.)

If the number of cones be indefinitely increased, the area of the bases of all the cones will ultimately be the area of the surface of the sphere.

∴ Volume of sphere = area of surface of sphere $\times \frac{r}{3}$

But area of surface of sphere = $4\pi r^2$

$$\therefore \text{Volume of sphere} = 4\pi r^2 \times \frac{r}{3} = \frac{4\pi r^3}{3}$$

RULE.—**Multiply the cube of the radius by four-thirds of π.**

NOTE.—Four-thirds of π = 4·1888.

(*h*) To find the volume of a sphere from that of the circumscribing cylinder.

Volume of cylinder $= 2\pi r^3$ (Sect. XXIX.)

Volume of sphere $= \frac{4\pi r^3}{3}$

$\therefore$ Volume of sphere : volume of circumscribing cylinder $:: \frac{4\pi r^3}{3} : 2\pi r^3$

$\therefore$ Volume of sphere $= \frac{2}{3}$ volume of circumscribing cylinder

RULE.—**Take two-thirds of the volume of the circumscribing cylinder.**

(*i*) Volumes of spheres are as the cubes of their radii.

Let there be two spheres whose radii are R and r.

Then volume of first sphere $= \frac{4\pi R^3}{3}$

And " second sphere $= \frac{4\pi r^3}{3}$

$\therefore$ Volume of first : volume of second $:: \frac{4\pi R^3}{3} : \frac{4\pi r^3}{3} :: R^3 : r^3$

(*k*) To find the volume of a spherical shell.

Let D and d = the outer and inner diameters.

Then R and $r = \frac{D}{2}$ and $\frac{d}{2}$

Volume of the shell will equal the difference of the volumes of the spheres whose diameters are D and d.

$$\therefore \text{Volume of shell} = \frac{4\pi D^3}{3 \cdot 2^3} - \frac{4\pi d^3}{3 \cdot 2^3} = \frac{4\pi D^3}{3 \times 8} - \frac{4\pi d^3}{3 \times 8}$$

$$= \frac{\pi D^3}{6} - \frac{\pi d^3}{6} = \frac{\pi}{6}(D^3 - d^3)$$

But $D^3 - d^3 = (D - d)(D^2 + Dd + d^2)$

$$\therefore \text{Volume of shell} = \frac{\pi}{3}\left(\frac{D - d}{2}\right)(D^2 + Dd + d^2)$$

That is, the volume of a shell equals the volume of a frustum of a cone whose ends are D and d respectively in radius, and whose height is the thickness of the shell (see Sect. XXXV.).

RULE.—**To the sum of the squares of the outer and inner diameters add the product of the two diameters, then multiply this result by one-half the difference of the two diameters multiplied by one-third of π.**

NOTE.—One-third of $\pi = 1{\cdot}0472$.

(l) To find the volume of a sphere, the diameter given.

$$\text{Volume of sphere} = \frac{4\pi r^3}{3} = \frac{8\pi r^3}{6} = \frac{\pi(2r)^3}{6} = \frac{\pi d^3}{6}$$

RULE.—**Multiply the cube of the diameter by one-sixth of π.**

NOTE.—One-sixth of $\pi = {\cdot}5236$.

(*See* EASY EXERCISES XV.)

EXERCISE XXXIII.

1. Find the surface of a sphere whose radius is $3\frac{1}{2}$ ft.
2. What is the radius of a sphere whose surface is 120 sq. ft.?
3. Find the volume of a sphere whose radius is 2 ft. 6 in.
4. Find the radius of a sphere whose volume is 3 cub. ft.
5. Find the volume of a sphere whose surface is 100 sq. in.
6. How many cubic inches of iron will be required to form a garden roller which is $\frac{1}{2}$ in. thick with an outer circumference of $5\frac{1}{2}$ ft. and a length of $3\frac{1}{2}$ ft.?
7. Find the surface of a sphere 25 in. in diameter.
8. Find the radius of a sphere which contains exactly one cubic yard.
9. What is the weight of a hollow sphere of metal whose inside diameter is 18 in., and thickness 2 in., given $\pi = 3\frac{1}{7}$, and that a cubic foot of metal weighs 7776 ozs.?
10. How much space will be left vacant after filling a box whose capacity is 1 cub. ft. with marbles each one inch in diameter?
11. A solid metal sphere 6 in. in diameter is formed into a tube 10 in. in external diameter and 4 in. in length: find the thickness of the tube.
12. How much of the earth's surface would a man see if he were raised to the height of the radius above it?
13. If the ball at the top of St. Paul's is 6 ft. in diameter, what will the gilding of it cost at $3\frac{1}{2}d.$ per square inch?
14. If the diameter of the earth be 8000 miles, and geologists know the interior to the depth of 5 miles below the surface, what fraction of the whole contents would be known?
15. A cubic foot of copper is drawn into a wire one-fourth of an inch in diameter: find its length.

16. A spherical shell whose circumference is 2 ft. weighs $\frac{19}{20}$ of a solid sphere of the same size and material: find the diameter of the internal cavity.

17. At a distance of 50 miles from a tower its top just appeared above the horizon: find its height, having given the earth's diameter, 7964 miles.

18. The weights of two globes are as 16 : 25, the weights of the substances composing the globes are as 15 : 9: compare the diameter of the globes.

19. If a shot $8\frac{3}{4}$ lbs. weight has a diameter of 3·96 in., find the weight of a shot whose diameter is 4·4 in.

20. The whole surface of a globe is 78·54 sq. ft.: find its circumference.

21. A sphere and a cube have the same surface: compare their volumes.

22. A cone of silver, whose base is 18 in. in diameter and whose height is $4\frac{1}{2}$ in., is melted and made into a spherical ball: find the diameter of the ball.

23. A sphere and a cube have the same volume: compare their surfaces.

24. The volume of a sphere is 11·754 cub. in.: find its surface.

25. A cylinder 1 ft. 6 in. deep, and 2 ft. 6 in. in diameter, is filled with water: find the diameter of a hemispherical basin which will hold 120 times as much water.

26. The surface of a sphere is three times as great as that of another sphere: compare their volumes.

27. A hollow sphere holds a litre: find the internal radius of the hollow sphere. A litre = a cubic decimetre, and a decimetre = $\frac{1}{10}$ of a metre (39·37 in.).

28. If the volumes of the earth, moon, and sun are proportional to the numbers 1, $\frac{27}{1331}$, and 1404928, express the surfaces of the moon and sun if that of the earth be taken as unity.

XXXIX.—ON SPHERICAL SECTORS, SEGMENTS, AND ZONES.

(*a*) To find the surface of a spherical sector, segment, or zone.

From the chapter on the sphere the surfaces of sectors, segments, and zones can be obtained from the following:—

$$\text{Surface} = 2\pi r \times h = 2\pi rh \quad \text{(Sect. XXXVIII.)}$$

RULE.—**Multiply the product of the radius of the sphere and the height of the sector, segment, or zone by twice π.**

(*b*) To find the volume of a spherical sector.

Let ABCD be a spherical sector.
„ RA = h = height of the arc.
„ CA = r = radius of the sphere.

From reasoning similar to that in Sect. XXXVIII. (g) for finding the volume of a sphere, it can be shown that the volume of a sector = area of spherical surface $\times \frac{r}{3}$

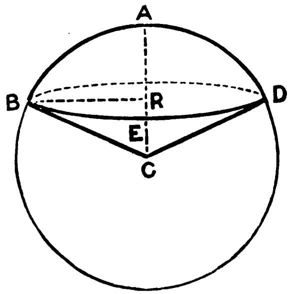


Fig. 1.

But area of spherical surface $= 2\pi r \times h$

$$\therefore \text{ Volume of sector } = 2\pi rh \times \frac{r}{3}$$

$$= \frac{2\pi r^2 h}{3} \quad \text{(i.)}$$

$$\text{But } 2\pi r = c$$

$$\therefore \text{ Volume of sector } = \frac{crh}{3} \quad \text{(ii.)}$$

Rule I.—**Multiply the product of the square of the radius and the height of the arc of the sector by two-thirds of π.**

Rule II.—**Multiply the product of the circumference and radius of the sphere by one-third of the height of the arc of the sector.**

(*c*) To find the volume of a spherical segment.

Let ABED (Fig. 1) be a spherical segment.

It is evident that the solidity of the segment ABED will be equal to the solidity of the sector ABCD less the solidity of the cone BCD.

Let diameter of sphere = D = $2r$.

Let height of segment = AR = h.

$$\text{Then solidity of sector ABCD } = 2\pi rh \times \frac{r}{3} = \frac{2\pi r^2 h}{3}$$

$$\text{And solidity of cone BCD } = \frac{\pi BR^2 \times CR}{3} \quad \text{(Sect. XXXIII.)}$$

Now BR = radius of the base of the cone.

And CR = height of the cone.

$$\text{But CR } = r - h$$

$$\text{And BR } = \sqrt{r^2 - (r - h)^2}$$

$$= \sqrt{r^2 - (r^2 - 2rh + h^2)} = \sqrt{2rh - h^2}$$

$$\therefore \text{ Solidity of cone } = \frac{\pi}{3}(2rh - h^2)(r - h)$$

$$\therefore \text{Solidity of segment} = \frac{2\pi r^2 h}{3} - \frac{\pi}{3}(2rh - h^2)(r - h)$$

$$= \frac{2\pi r^2 h}{3} - \frac{\pi}{3}(2r^2 h - 3rh^2 + h^3)$$

$$= \frac{2\pi r^2 h}{3} - \frac{2\pi r^2 h}{3} + \pi r h^2 - \frac{\pi h^3}{3}$$

$$= \pi r h^2 - \frac{\pi h^3}{3} = \pi h^2\left(r - \frac{h}{3}\right) \quad \text{(i.)}$$

$$\text{But } r = \frac{D}{2}$$

$$\therefore \text{Solidity of segment} = \pi h^2\left(\frac{D}{2} - \frac{h}{3}\right) = \pi h^2\left(\frac{3D - 2h}{6}\right)$$

$$= \frac{\pi h^2}{6}(3D - 2h) \quad \text{(ii.)}$$

Let $BR = b$ = radius of the base of the segment.

$$\text{Because } BR^2 = 2rh - h^2$$

$$\therefore r = \frac{BR^2 + h^2}{2h} = \frac{b^2 + h^2}{2h}$$

$$\text{But solidity of segment} = \pi h^2\left(r - \frac{h}{3}\right) \quad \text{(See (i.) above.)}$$

$$\therefore \text{Solidity of segment} = \pi h^2\left(\frac{b^2 + h^2}{2h} - \frac{h}{3}\right)$$

$$= \pi h^2\left(\frac{3b^2 + 3h^2 - 2h^2}{6h}\right)$$

$$= \frac{\pi h}{6}(3b^2 + h^2) \quad \text{(iii.)}$$

Rule I.—**From the radius of the sphere subtract one-third of the height of the segment, and multiply this result by the square of the height of the segment multiplied by π.**

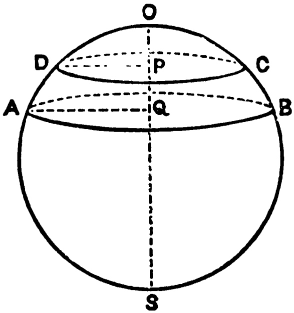
Rule II.—**From three times the diameter of the sphere take twice the height of the segment, and multiply this result by the square of the height multiplied by one-sixth of π.**

Rule III.—**To three times the square of the radius of the base of the segment add the square of the height of the segment, and multiply the result by the height of the segment multiplied by one-sixth of π.**

(*d*) To find the volume of a spherical zone.

Let the sphere be cut by two parallel planes AB and CD.
It is required to find the volume of the spherical zone ABCD.

It is evident that the volume of the spherical zone ABCD equals the difference between the volumes of the spherical segments ABO and DCO.



Let r_1 = AQ = radius of the base of the segment ABO.

Let h_1 = OQ = height of the segment ABO.

Let r_2 = DP = radius of the base of the segment DCO.

Let h_2 = OP = height of the segment DCO.

Volume of segment ABO = V_1

$$= \frac{\pi h_1}{6}(3r_1^2 + h_1^2)$$

Volume of segment DCO = V_2

$$= \frac{\pi h_2}{6}(3r_2^2 + h_2^2)$$

$$\therefore \text{Volume of zone ABCD} = V = V_1 - V_2$$

$$= \frac{\pi h_1}{6}(3r_1^2 + h_1^2 - \frac{\pi h_2}{6}(3r_2^2 + h_2^2)$$

$$\therefore V = \frac{\pi}{6}\{h_1(3r_1^2 + h_1^2) - h_2(3r_2^2 + h_2^2)\}$$

$$= \frac{\pi}{6}(h_1 3r_1^2 + h_1^3 - h_2 3r_2^2 - h_2^3)$$

$$= \frac{\pi}{6}(h_1 3r_1^2 - h_2 3r_1^2 - h_2 3r_2^2 + h_1 3r_2^2 + h_1^3 - h_2^3 + h_2 3r_1^2 - h_1 3r_2^2)$$

by adding and subtracting $h_2 3r_1^2 - h_1 3r_2^2$

$$= \frac{\pi}{6}\{(h_1 - h_2)3r_1^2 - (h_2 - h_1)3r_2^2 + h_1^3 - h_2^3 + 3r_1^2 h_2 - 3r_2^2 h_1\} \quad \text{(i.)}$$

Let R = radius of the sphere.

Then PS = $2R - h_2$

And QS = $2R - h_1$

Now $OP \cdot PS = DP^2 = r_2^2$

$$\therefore h_2(2R - h_2) = r_2^2 \qquad \therefore 2Rh_2 - h_2^2 = r_2^2 \qquad \therefore 2R = \frac{r_2^2 + h_2^2}{h^3}$$

Again $OQ \cdot QS = AQ^2 = r_1^2$

$$\therefore h_1(2R - h_1) = r_1^2 \qquad \therefore 2Rh_1 - h_1^2 = r_1^2 \qquad \therefore 2R = \frac{r_1^2 + h_1^2}{h_1}$$

$$\therefore \frac{r_1^2 + h_1^2}{h_1} = \frac{r_2^2 + h_2^2}{h_2} \qquad \therefore r_1^2 h_2 + h_1^2 h_2 = r_2^2 h_1 + h_2^2 h_1$$

$$\therefore r_1^2 h_2 - r_2^2 h_1 = h_2^2 h_1 - h_1^2 h_2$$

$$\therefore 3r_1^2 h_2 - 3r_2^2 h_1 = 3h_2^2 h_1 - 3h_1^2 h_2$$

Substitute this value of $3r_1^2 h_2 - 3r_2^2 h_1$ in eq. (i.)

$$\text{Then } V = \frac{\pi}{6}\{(h_1 - h_2)3r_1^2 - (h_2 - h_1)3r_2^2 + h_1^3 - h_2^3 + 3h_2^2h_1 - 3h_1^2h_2\}$$

$$= \frac{\pi}{6}\{(h_1 - h_2)3r_1^2 - (h_2 - h_1)3r_2^2 + (h_1 - h_2)^3\}$$

$$= \frac{\pi}{6}\{(h_1 - h_2)3r_1^2 + (h_1 - h_2)3r_2^2 + (h_1 - h_2)^3\}$$

$$= \frac{\pi}{6}(h_1 - h_2)\{3r_1^2 + 3r_2^2 + (h_1 - h_2)^2\}$$

But $h_1 - h_2 = h =$ height of zone

$$\therefore V = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$$

RULE.—**Multiply the sum of the squares of the radii of the two ends by three, then add the square of the height of the zone, and multiply the result by the height of the zone multiplied by one-sixth of π.**

(e) From the formula $\frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$—that is, the volume of a zone —to obtain the volume of a segment of a sphere and of a sphere.

If $r_2 = 0$, the zone becomes a segment, and volume of segment $= \frac{\pi h}{6}(3r_1^2 + h^2)$, where $h =$ height of the segment.

If $r_1 = 0$, and $r_2 = 0$, the zone becomes a sphere, and volume of sphere $= \frac{\pi h}{6} \times h^2 = \frac{\pi h^3}{6}$, where $h =$ diameter of the sphere.

These are the same results as were obtained for the volumes of the sphere and segment respectively in Sects. XXXVII. and XXXVIII.

EXERCISE XXXIV.

1. Find the surface of the segment of a sphere which is 6 in. in diameter, and the height of the segment 2 in.

2. Find the convex surface of a slice 2 ft. high, cut off from a globe 17 ft. in radius.

3. What is the height of the segment of a sphere whose convex surface is 100 sq. in., the radius of the sphere being 3 ft.?

4. If the pressure of the atmosphere at the earth's surface be 15 lbs. to the square inch, how much weight of atmosphere does the earth support? Take the radius of the earth as r inches.

5. Find the surface of the segment of a sphere 2 ft. high, cut off from a globe of 12 ft. radius.

6. Find the volume of a segment of a sphere 10 ft. high, the radius of the sphere being 20 ft.

7. The height of a spherical segment is 6 ft., and the circumference of its base 20 ft.: find its volume.

8. In a sphere whose radius is 3 ft., find the height of a segment which has one-third of the surface of the sphere.

9. From what height above the surface of the earth will one-fifth of the surface be seen? Diameter of the earth = 8000 miles.

10. The height of a zone of a sphere is $2\frac{1}{2}$ ft., and the diameter of the sphere is $6\frac{1}{2}$ ft.: find the area of the curved surface.

11. Find the area of the surface of a zone of a sphere whose circumference is 32 ft., and the distances of the ends of the zone from the centre 3 ft. and 2 ft. on opposite sides of the centre.

12. Find at what distance from the surface of a sphere an eye must be placed to see one-sixteenth of the surface. Diameter of the earth = 8000 miles.

13. At what distance is the top of the peak of Teneriffe, which is 11,000 ft. high, visible just above the horizon, the earth's diameter being 7964 miles?

14. From a sphere whose radius is 3 ft. a segment is cut off whose convex surface is 30 sq. ft.: find the surface of its base.

15. Find the radius of the base of a cone which has the same volume as a sphere whose radius is 5 ft., and the height of the cone one-half the radius of the sphere.

16. Find the radius of a sphere which has the same volume as a cone whose height is 10 ft., and radius of base 2 ft.

17. A sphere is 90 ft. in diameter: at what distance from the surface will $\frac{4}{33}$ of its entire surface be seen?

18. Find how much of the surface of a sphere will be seen by a person placed $\frac{1}{7}$ of the radius of the sphere from the surface.

19. Find the volume of a spherical zone, the radii of the two ends being 3 ft. and 2 ft. respectively, and the height of the zone $1\frac{1}{2}$ ft.

20. Find the volume of a spherical segment of a sphere whose radius is 4 ft., the height of the segment being $2\frac{1}{2}$ ft.

21. Find the volume of a spherical segment, the radius of whose base is 2 ft., and height $1\frac{1}{2}$ ft.

XL.—EXAMPLES OF THE SOLUTION OF A FEW PROBLEMS.

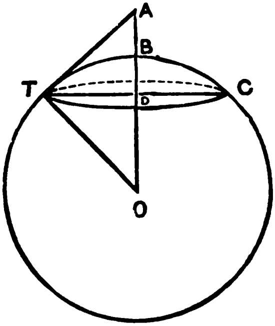
1. What part of the whole surface of a sphere will be seen from a height of 5 miles?

Let AB = 5 miles.

Let AT = tangent from A.

It is required to find the surface of the segment TBC.
To find BD, the height of the segment.
Let BD = h. $\therefore$ OD = $r - h$.
Now ATO and TDO are similar triangles.

$$\therefore \text{TO} : \text{OD} :: \text{OA} : \text{OT}$$
$$\therefore r : r - h :: r + 5 : r$$
$$\therefore r^2 = (r - h)(r + 5)$$
$$\therefore r^2 = r^2 - rh + 5r - 5h$$



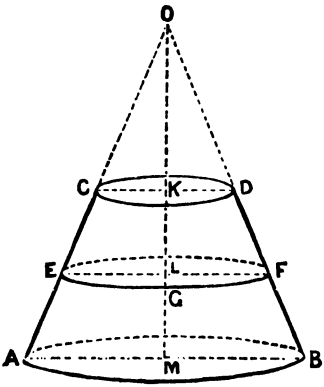
$$\therefore rh + 5h = 5r \qquad \therefore h = \frac{5r}{r+5}$$

But surface of sphere : surface of segment :: $2r : h :: 2r : \frac{5r}{r+5}$

$\therefore$ Surface of segment

$$= \frac{5r}{2r(r+5)} \text{ of surface of sphere}$$
$$= \frac{5}{2(r+5)} \text{ of surface of sphere}$$
$$\frac{5}{2(r+5)} \text{ part} \quad \text{Ans.}$$

2. The diameters of the ends of a frustum of a cone are 4 ft. and 6 ft., and the height of the frustum is 3 ft. It is required to divide the frustum into two equal parts by a plane parallel to the base.



Let ABCD be the frustum.

Let EGF be the plane parallel to the base dividing it into two equal portions.

Let MK = height of frustum.

It is required to find LM, the distance of the dividing plane from AB the base.

Complete the cone OAB, of which ABDC is the frustum.

$$\text{Then OK} : \text{KC} :: \text{OM} : \text{AM}$$
$$\text{OK} : 2 :: \text{OK} + 3 : 3$$
$$\therefore 3\text{OK} = 2\text{OK} + 6$$
$$\therefore \text{OK} = 6, \text{ and OM} = 9$$

Now, frustum ABEF = cone OAB − cone OEF

And frustum EFDC = cone OEF − cone OCD

But frustum ABEF = frustum EFDC

$$\therefore \text{Cone OAB} - \text{cone OEF} = \text{cone OEF} - \text{cone OCD}$$
$$\therefore 2 \text{ cone OEF} = \text{cone OAB} + \text{cone OCD}$$
$$\therefore \text{Cone OEF} = \tfrac{1}{2}(\text{cone OAB} + \text{cone OCD})$$

But these cones are similar solids

$\therefore$ They are as the cubes of like dimensions.

Height of OEF = OL, height of OAB = 9, height of OCD = 6

$\therefore$ Cone OEF : $\frac{1}{2}$(cone OAB + cone OCD) :: $OL^3 : \frac{1}{2}(9^3 + 6^3)$

But cone OEF = $\frac{1}{2}$(cone OAB + cone OCD)

$\therefore OL^3 = \frac{1}{2}(9^3 + 6^3) = 472\cdot5$

$\therefore OL = 7\cdot8$ nearly

$\therefore$ The plane EGF must be cut $9 - 7\cdot8 = 1\cdot2$ ft. from the base AB. Ans.

3. There are three similar rectangular solids which are respectively 4 ft., 5 ft., and 6 ft. in length. The volume of the largest is 30 cub. ft. greater than one-third of the capacity of the other two. Find the volume of each of the solids.

The volumes are as the cubes of the lengths.

Let S_1, S_2, S_3 = the largest, mean, and least solids in volume.

Then $S_1 : S_2 : S_3 :: 6^3 : 5^3 : 4^3 :: 216 : 125 : 64$

$$\text{Now } S_1 : \frac{S_2 + S_3}{3} :: 216 : \frac{125 + 64}{3} :: 216 : 63 :: 24 : 7$$

That is, the largest is $\frac{17}{24}$ of itself larger than one-third of the sum of the volumes of the other two solids.

But $\frac{17}{24}$ of itself = 30 cub. ft.

That is, $\frac{17}{24}S_1$ = 30 cub. ft.

$$\therefore S_1 = \frac{30 \times 24}{17} = 42\tfrac{6}{17}$$

But $S_1 : S_2 :: 216 : 125$

$$\therefore S_2 = \frac{S_1 \times 125}{216} = \frac{42\frac{6}{17} \times 125}{216} = 24\tfrac{26}{51}$$

And $S_1 : S_3 :: 216 : 64$

$$\therefore S_3 = \frac{S_1 \times 64}{216} = \frac{42\frac{6}{17} \times 64}{216} = 12\tfrac{28}{51}$$

$42\frac{6}{17}$ cub. ft., $24\frac{26}{51}$ cub. ft., and $12\frac{28}{51}$ cub. ft. Ans.

XLI.—ARTIFICERS' MEASUREMENTS.

Various methods are used by workmen in measuring their work. The standard for length is the foot or yard; the standard for area, the square foot or square yard; and for solidity, the cubic foot or cubic yard.

I. Flooring, roofing, tiling, and plastering are usually estimated for by the number of squares of 10 ft., or 100 sq. ft., contained in the area to be covered.

The area of the roof can be found from the base measurements if the pitch of the roof is known. The pitch is the ratio of the perpendicular height of the roof to the breadth of the span.

There are three pitches in most common use.

(*a*) *The common pitch*, where the length of the rafters meeting at the apex each equal three-quarters of the breadth of the building.

$$\text{Therefore area of roof} = \text{twice } (\tfrac{3}{4} \text{ breadth} \times \text{length})$$
$$= 1\tfrac{1}{2} \text{ times area of base of building}$$

(*b*) *The Gothic pitch*, where the length of the rafters each equal the breadth of the building.

$$\text{Therefore area of roof} = \text{twice (breadth} \times \text{length)}$$
$$= \text{twice area of the base of building}$$

(*c*) *The pediment pitch*, where the ratio of the perpendicular height of the roof is to the breadth of the span as 2 : 9.

$$\text{Hence length of rafters} = \sqrt{2^2 + (\tfrac{9}{2})^2} = \tfrac{9 \cdot 84}{2} = 4 \cdot 92$$
$$\therefore \text{Length of rafters} : \text{breadth of building} :: 4 \cdot 92 : 9 = \tfrac{492}{900} = \tfrac{55}{100} \text{ nearly}$$
$$= \tfrac{11}{20} \text{ nearly}$$
$$\therefore \text{Area of roof} = \text{twice } (\tfrac{11}{20} \text{ of breadth} \times \text{length})$$
$$= \tfrac{11}{10} \text{ of area of base of building}$$

II. The standard measure for brickwork is a wall of brickwork which has a surface of a square rod, or $272\tfrac{1}{4}$ sq. ft., and is one brick and a half thick.

To find the number of standard rods of brickwork in a wall, it will be necessary first to divide the area of the wall in square feet by $272\tfrac{1}{4}$; this will be the number of rods if the wall is one and a half bricks thick: but if the wall be of another thickness, multiply this number of rods by the number of half-bricks in the thickness, and divide by three.

Example.—Find the number of standard rods of brickwork in a wall 250 ft. long, 10 ft. high, and $2\tfrac{1}{2}$ bricks thick.

$$\frac{250 \times 10}{272\tfrac{1}{4}} = \text{number of rods } 1\tfrac{1}{2} \text{ brick thick}$$

$$\frac{250 \times 10}{272\tfrac{1}{4}} \times \frac{5}{3} = \text{number of rods } 2\tfrac{1}{2} \text{ bricks thick}$$

$$\frac{250 \times 10 \times 4 \times 5}{1089 \times 3} = \frac{50000}{3267} = 15 \cdot 3 \quad \text{Ans.}$$

A common brick, with the mortar used for laying it, is taken as 9 in. long, $4\tfrac{1}{2}$ broad, and 3 in. deep.

The number of bricks required to make a standard rod of brickwork is obtained by dividing the solid contents of a rod of brickwork by the volume of one brick.

$$\text{Thus } 272\tfrac{1}{4} \times 1\tfrac{1}{8} \div (\tfrac{3}{4} \times \tfrac{3}{8} \times \tfrac{1}{4}) = \tfrac{1089}{4} \times \tfrac{9}{8} \times \tfrac{128}{9}$$
$$= 1089 \times 4 = 4356 \text{ bricks}$$

A percentage must be allowed for waste, therefore it is generally taken that a standard rod of brickwork requires 4500 bricks, when waste is allowed for.

III. Logs of timber are usually measured by the rules given in the previous sections for the various solids which the logs of timber resemble in shape. In practice, though, simpler rules are substituted, which in a few cases, although not exact, are easily applied.

The volume of four-sided or squared timber is usually found by multiplying the mean thickness by the mean breadth, and that by the length.

The mean thickness or breadth is found by taking several measurements at equidistant points and dividing the sum of these measurements by the number of measurements.

Example.—The length of a piece of timber is 24 ft., and breadth $4\frac{1}{2}$ ft., whilst the thicknesses at various equidistant points are 1 ft., 1 ft. 3 in., 1 ft. 9 in., and 2 ft.: find the volume.

$$\text{Here the mean thickness} = \frac{1 + 1\frac{1}{4} + 1\frac{3}{4} + 2}{4} = 1\tfrac{1}{2} \text{ ft.}$$

$$\therefore \text{Volume} = 24 \times 4\tfrac{1}{2} \times 1\tfrac{1}{2} = 24 \times \tfrac{9}{2} \times \tfrac{3}{2} = 162 \text{ cub. ft. Ans.}$$

The volume of round or unsquared timber is found by multiplying the square of the mean quarter girth by the length.

Example.—The length of a piece of timber just felled is 54 ft. The equidistant girths are $3\frac{1}{2}$ ft., $4\frac{1}{2}$ ft., 5 ft., $5\frac{1}{2}$ ft., and $6\frac{1}{2}$ ft.: find the volume.

$$\text{Here the mean girth} = \frac{3\frac{1}{2} + 4\frac{1}{2} + 5 + 5\frac{1}{2} + 6\frac{1}{2}}{5} = 5 \text{ ft.}$$

$$\therefore \text{Mean quarter girth} = \tfrac{5}{4} \text{ ft.}$$

$$\therefore \text{Volume} = (\tfrac{5}{4})^2 \times 54 = \tfrac{25}{16} \times 54 = \tfrac{675}{8} = 84\tfrac{3}{8} \text{ cub. ft. Ans.}$$

IV. Painters usually estimate their work by the square yard, and glaziers by the square foot.

Plumbers' work is charged by the cwt. or lb. for sheet lead. Lead piping is charged by the foot run, and varies according to diameter and thickness.

EXERCISE XXXV.

1. Required the cost of lining a water-cistern 3 ft. long, 2 ft. deep, and 2 ft. 6 in. broad, with sheet lead of 10 lbs. per square foot at 40*s.* a cwt.

2. Find the cost of painting a door on both sides, which measures $7\frac{1}{4}$ ft. by $3\frac{3}{4}$ ft., at 9*d.* a square yard.

3. Find the charge for plastering a room 18 ft. long, 15 ft. broad, and 9 ft. high, the walls at 5*d.* a square yard, and the ceiling at 11*d.* per square yard, allowing for a door 7 ft. by 4 ft., and a fireplace 4 ft. 6 in. by 4 ft.

4. Find the cost of a wall with a triangular gable top of 10 ft. high, the height of the wall being 36 ft., the breadth 24 ft., and the thickness 2 bricks, at 34*s.* per standard rod.

5. Find the cost of roofing a house of the common pitch, the length being 40 ft., and the breadth 35 ft., at 12*s.* a square.

6. If bricks are £2 5*s.* a thousand, and 4500 are required for a standard rod, mortar and cartage cost 24*s.* 9*d.* a rod, and labour £2 5*s.* per rod, find the cost of building a wall 272 ft. long, 18 ft. high, and 2 bricks thick.

7. A piece of timber which tapers regularly has one end 7 ft. in girth, the other end 4 ft., and is 40 ft. long: find its volume in cubic feet.

8. A building has 63 windows; 40 of them contain 12 panes each, 20 in. by 16 in.; the others contain 9 panes each, 16 in. square: find the cost of glazing the whole at 2*s.* 3*d.* per square foot.

9. Find the cost of flooring a room 54 yds. long and 21 yds. broad, with planks each $13\frac{1}{2}$ ft. long and $10\frac{1}{2}$ in. wide, at $5\frac{1}{2}$*d.* per square foot. Find also the number of planks required.

10. The trunk of a tree when felled is 40 ft. long, and the girth of its ends is 7 ft. and 4 ft.: what is its value at 1*s.* per cubic foot?

11. The length of a house is 80 ft., and its breadth 35 ft.: find the cost of roofing it, common pitch, at 7*s.* 6*d.* a square.

12. A piece of squared timber 36 ft. long tapers regularly; at one end its breadth and thickness are 30 in. and 20 in. respectively, and at the other 24 in. and 18 in.: find the number of cubic feet.

13. Find the number of standard rods of brickwork in a wall 125 ft. long, 14 ft. 8 in. high, and $2\frac{1}{2}$ bricks thick.

14. Find the cost of roofing a building of the Gothic pitch at 30*s.* per square, the length of the building being 120 ft., and the breadth 40 ft.

15. The length of the roof of a house is 75 ft., and the length of a string stretched over the ridge from eaves to eaves is 60 ft.: find the cost of the roof at £2 7*s.* 6*d.* per square.

16. Find the cost of flooring two rooms at £3 15*s.* per square, one room measuring 28 ft. by 16 ft., and the other 24 ft. by 15 ft. 6 ins.

17. Find the number of cubic feet in a felled piece of timber whose equidistant girths are 3 ft., $4\frac{1}{2}$ ft., $5\frac{1}{2}$ ft., and 7 ft., the length being 32 ft.

18. A building 30 ft. long and 25 ft. broad is to be covered with lead, the roof being of pediment pitch: find the cost of the lead, supposing it to weigh 6 lbs. to the square foot and to cost 21*s*. per cwt.

19. Find the number of yards of standard brickwork contained in a triangular gable top which is 15 ft. high, and the base of which is 20 ft., supposing the thickness of the wall to be $2\frac{1}{2}$ bricks.

20. What is the cubical measurement in feet of a squared piece of timber whose length is 36 ft., and mean breadth and thickness 1 ft. 3 in. and 9 in. respectively?

21. What number of cubic feet is there in the trunk of a tree recently felled which is 48 ft. in length, and whose equidistant girths are $2\frac{1}{2}$ ft., $3\frac{1}{2}$ ft., $4\frac{1}{2}$ ft., and $5\frac{1}{2}$ ft.?

22. How many standard rods of brickwork are there in a wall 78 yds. long, 8 ft. high, and $2\frac{1}{2}$ bricks thick?

23. What will it cost to roof a Gothic house which is 55 ft. by 40 ft., at 25*s*. a square?

24. What will be the cost of glazing 18 windows, each containing 12 panes of 16 in. by 12 in., at 2*s*. 6*d*. a square foot?

25. Find the cost of lining a cistern with lead, 5 lbs. to the square foot, at 24*s*. a cwt., the cistern being 8 ft. long, 4 ft. broad, and 3 ft. deep.

XLII.—MISCELLANEOUS EXERCISES ON SOLIDS.

I.

1. Find to two decimal places the edge of a cube equal in volume to a sphere of 4 ft. radius.

2. The height of a cone is 2·712 in., and its volume is 27·12 cub. in.: find to three decimal places the circumference of the base.

3. A brick wall is half a mile long, 10 ft. high, $4\frac{3}{4}$ ft. thick at the bottom, and tapering uniformly to 3 ft. thick at the top: how many cubic yards of brickwork does it contain?

4. A circular plate of iron weighs $4\frac{3}{4}$ ozs. per cubic inch; it is $14\frac{1}{8}$ in. in diameter, and 2 in. thick: find its whole weight in pounds. $\pi = \frac{355}{113}$.

5. Find the cost of painting the convex surfaces of 5 cylindrical pillars, each 14 ft. high and a foot in diameter, at $8\frac{1}{2}$*d*. a square yard.

6. A regular tapering plank 7 ft. long, $\frac{7}{8}$ of an inch thick, is 8 in. wide at one end and 5 in. wide at the other: find its contents.

II.

1. Find to the hundredth part of an inch the slant height of a cone 58·9 cub. ft. in volume and 5 ft. in diameter at the base.

2. The diameter of a sphere is 6 ft.: how many cubic feet of it must be removed that the remainder may form the largest cube that can be cut from it?

3. A room 11 ft. high is half as long again as it is wide, and its cubical contents are $4768\frac{1}{2}$ cub. ft.: find its length and breadth.

4. Find the weight of a circular disc of cast iron 7 ft. in diameter and $1\frac{1}{3}$ in. thick, if a plate 1 ft. square and 1 in. thick weighs $37\frac{1}{2}$ lbs., and the ratio of the diameter to the circumference be as 7 : 22.

5. The inner diameter of a cylindrical cistern is 3 ft. 9 in.; its depth is $5\frac{3}{5}$ ft.: how many gallons of water will it hold?

6. Find the volume of Pompey's pillar, the diameter of whose base is 9 ft., and of the summit $7\frac{1}{2}$ ft., the vertical height being 90 ft.

III.

1. The height of a cone is 20 in., the circumference at the base 72 in.: find its volume.

2. At 2*s.* 6*d.* a square foot, what is the cost of polishing the convex surface of a cylinder $10\frac{1}{2}$ in. in diameter and 8 ft. long?

3. If the volumes of two cylinders are as 11 : 8, and their heights as 3 : 4, and if the base of the first has an area of 16·5 sq. ft., what is the area of the base of the second?

4. From what height above the surface of the earth will one-third of its surface be seen?

5. What number of gallons of water will a tank $10\frac{1}{2}$ ft. long, $6\frac{1}{8}$ ft. wide, and 5 ft. deep hold?

6. The weights of two spheres which are solid and made of the same material are 512 lbs. and 729 lbs. respectively: if the radius of the first sphere is 16 in., what will it cost to gild the surface of the second sphere at $1\frac{3}{4}$*d.* a square inch? Given $\pi = \frac{22}{7}$.

IV.

1. The capacity of a cylindrical vat is to be 7000 gals., and its depth 9 ft. 5 in.: what must be the internal diameter? $\pi = \frac{355}{113}$.

2. What is the length of the diagonal of a rectangular parallelopiped whose length is 6 ft., breadth 4 ft., and depth 2 ft.?

3. The perimeter of a circle is 1 ft.: find its area to the hundredth of a square inch.

4. Find the volume of an hexagonal pyramid, each side of the base being 6 ft., and the perpendicular height 8 ft.

5. The circumference of the earth being 25,000 miles, and the distance

between London and York 200 miles, to what height must a man ascend from one of these places in order to see the other?

6. The volume of a sphere is equal to that of a cube the length of whose edge is 1 ft.: find the surface of each.

V.

1. The three edges of a rectangular parallelopiped that meet at an angle are respectively 25 ft., 54 ft., and 160 ft.: find the diagonal of a cube which has the same volume.

2. A tank is 20 ft. 9 in. long, 15 ft. 7 in. wide, and 6 ft. 4 in. deep: find how much water it will hold in cubic feet.

3. How many gallons of water will be required to fill a cylindrical tank 20 ft. 2 in. deep and 21 ft. in diameter?

4. A right cone and a hemisphere lie on opposite sides of a common base of 2 ft. diameter; the cone is right-angled at the vertex: if a cylinder circumscribe them in this position, how much additional space is thereby enclosed?

5. To what height must a man be raised above the earth in order that he may see one-sixth part of its surface?

6. The volume of a sphere is equal to that of a right circular cylinder, the radius of the base of which is 1 ft., and the height 2 ft.: find the surface of each.

VI.

1. If I pay one guinea for a cubical block of marble of which the side measures 1 ft., what ought I to pay for another cubical block of the same marble of which the side is equal in length to the diagonal of the first block?

2. Find to the nearest hundredth of an inch the radius of a sphere whose volume is 1 cub. ft.

3. What must be the length of the side of a square tank capable of holding 2000 gals., if the depth of the tank is 5 ft.?

4. A cask in the form of two conic frustums joined at their bases has the diameter at the head 20 in. and at the bung 25 in.; the height of the cask is 3 ft. 4 in.: find the weight of water required to fill it.

5. In a spherical zone the radii of the two ends are 10 ft. and 6 ft.; the altitude of the zone is 8 ft.: find the volume.

6. The surface of a sphere is equal to that of a cube the length of whose edge is 1 ft.: find the volume of each.

VII.

1. If 30 cub. in. of gunpowder weigh 1 lb., what weight of gunpowder will be required to fill a cylinder which is 8 in. in internal diameter, and $2\frac{1}{2}$ ft. high?

2. An ale-glass in the form of a conic frustum is $3\frac{5}{8}$ in. in depth; the diameter of the top is $2\frac{1}{4}$ in., and that of the bottom 1 in.: find how many such glasses could be filled from a gallon.

3. Compare the surfaces of the three zones of the earth's hemisphere, the torrid zone extending to $23\frac{1}{2}°$ from the equator, and the frigid zone to $23\frac{1}{2}°$ from the pole, the temperate zone occupying the space between.

4. Find the volume of a cask in gallons, the length being 47·5 in., the bung diameter 28·6 in., and the head diameter 26·5 in.

5. The weights of two globes are as 9 : 25; the weights of the substances composing the globes are as 15 : 9: compare the diameters of the globe.

6. A cubic inch of brass is drawn into a wire $\frac{1}{25}$ of an inch diameter: find the length of the wire to the nearest inch.

VIII.

1. Water is poured into a cylindrical reservoir 20 ft. in diameter at the rate of 400 gals. a minute: if a gallon of water measures $277\frac{1}{4}$ cub. in., find how much the water rises in 1 min.

2. Find the surface of a triangular prism whose length is 5 ft., and the sides of whose base are 6 in., 8 in., and 10 in. respectively.

3. It is required to cut a piece equal to a solid foot from a plank $2\frac{1}{2}$ in. thick and 8 in. wide: find the length to be cut off.

4. The convex surface of a cone is 667·6 sq. ft., and the diameter of the base is 8·5 ft.: find the slant height.

5. What length of wire ·16 in. in diameter can be made out of 2 cub. in. of copper?

6. A sphere and a cube have the same volumes: compare their surfaces.

IX.

1. In a rectangular piece of ground 160 ft. long and 80 ft. broad, it is required to make a rectangular bath 145 ft. long and 65 ft. broad; the earth excavated is to be placed on the surrounding ground: what must be the depth of the bath in inches that the surrounding ground may be raised 8 ft.?

2. It is required to put a cubical case whose volume is 4019·679 cub. ft. through a square hatchway whose area is 37791·36 sq. in.: can this be done?

3. If Canadian elm is ·725 of the weight of water, what will be the weight of a beam of Canadian elm 12 ft. 6 in. long, 1 ft. 6 in. deep, and 1 ft. 3 in. thick?

4. If a heavy sphere 4 in. in diameter be placed in a conical glass full of water, whose diameter is 5 in. and altitude 6 in., find how many cubic inches of water will run over.

5. A mirror 33 in. by 22 in. is to have a frame the same area as that of the glass: find the width of the frame.

6. The solid contents of a box are $58\frac{7}{9}$ cub. ft.; the width is 5 ft. 4 in., the length is three times the depth: find the length and depth.

X.

1. What are the solid contents of a cylinder whose diameter is $4\frac{1}{2}$ ft., and height 8 ft.?

2. Find the volume of an hexagonal prism which is $25\frac{1}{4}$ ft. long, and the central diagonal of its base $2\frac{1}{2}$ ft.

3. Find the area of a circular ring whose internal and external diameters are 5 ft. and 15 ft. respectively.

4. The length of a hollow roller is 3 ft., the exterior diameter 2 ft., the thickness of the metal $\frac{3}{4}$ of an inch: find the solid contents.

5. A tank measuring 12 ft. by 8 ft. 10 in., and 6 ft. 6 in. deep, is filled with a liquid solution; after a deposit has taken place, the clear liquid is drawn off and found to measure 4000 gals. (gal. = 277·279 cub. in.): what is the value of the deposit at £2 3*s*. $11\frac{1}{2}$*d*. per cubic foot?

6. A globe 19 in. in diameter weighs 73 lbs.: what will be the weight of a globe made of the same material whose diameter is 38 in.?

XLIII.—MISCELLANEOUS EXERCISES (GENERAL).

(*See* Easy Exercises XVI. A, B, C, D, E, F, G, H.)

I.

1. Find the area of a circle traced on a sheet of paper by a pair of compasses whose legs are 6 in. long and contain an angle of 90°.

2. For one inch of rainfall, what is the weight of water on an acre of ground?

3. A railway embankment across a valley has the following measurements: width at the top, 20 ft.; at the base 45, ft.; height, 11 ft.; length at top, 1020 yds.; at the base, 960 yds.: find its cubic contents.

4. A lighthouse bears south-west from a ship $22\frac{1}{2}$ miles; the ship then sails due west for $13\frac{1}{2}$ miles, when she is due north of the lighthouse: how far off is the lighthouse now?

5. What is the value of a log of Spanish mahogany 18 ft. long, $3\frac{1}{8}$ ft. broad, and 2 ft. thick, at 7*s*. 6*d*. per cubic foot?

6. A well 3 ft. in diameter has a depth of water in it of 15 ft.: how many gallons of water are there in the well?

II.

1. A circular hole is to be cut in a circular plate whose diameter is 12 ft., so that the weight of the plate is reduced one-quarter: find the diameter of the hole.

2. A cubic foot of copper is drawn into a wire $\frac{1}{10}$ of an inch in diameter: find its length.

3. The distance between two towns is 54 miles, and their distances between the places on a map is $6\frac{3}{4}$ in.: find the scale to which the map is drawn.

4. A river 20 ft. deep, 100 yds. wide, flows at the rate of 3 miles an hour: find how many tons of water run into the sea per minute, if a cubic foot of water weighs 1000 ozs.

5. Find the volume of a cylinder whose radius is 4 ft., and height 100 ft.

6. A rectangular court is 20 yds. longer than it is broad, and its area is 4524 sq. yds.: find its length and breadth.

III.

1. A cube contains 11 cub. ft. 675 cub. in.: find the length of its diagonal.

2. A rectangular garden contains 1200 sq. yds., and the length is to the breath as 4 : 3: what will the fencing cost at 3*s.* 6*d.* per yard?

3. Find the size of the largest circular plate that can be cut out of a square plate of iron containing 25281 sq. in.

4. A wall five times as high as broad, and eight times as long as high, contains 18225 cub. ft.: find the breadth of the wall.

5. A hemispherical punch-bowl is 5 ft. 6 in. round the brim: supposing it to be half full, how many persons may be served from it in hemispherical glasses $1\frac{3}{4}$ in. in diameter at the top?

6. The radius of a circle is $\sqrt{2}$ in.; two parallel straight lines are drawn in it, each an inch from the centre: find the area of the part of the circle between the straight lines.

IV.

1. If two cubical blocks of stone contain 8 cub. ft., and the side of the lesser is to the greater as 3 : 4, find the side of each.

2. The area of a triangle is 6 ac. 2 ro. 8 po., and a perpendicular from one angle on the base measures 524 lks.: find the length of the base in chains.

3. The chord of an arc of a circle is 8 ft., and the height of the arc is 2 ft.: what is the radius of the circle?

4. Find the number of square inches in the surface of a glass shade in the shape of a cylinder with a hemispherical top, the diameter of the shade being 12 in., and the total height to the top of the dome 2 ft.

5. The map of a country is drawn on a scale of $\frac{1}{10}$ of an inch to a mile: what area on the map will represent a lake of 4000 acres?

6. The base of a rectangular prism is an equilateral triangle with a side of 7 in. The height of the prism is 2 ft.: find its volume.

V.

1. The sides of a right-angled triangular field adjacent to the right angle are 357 ft. and 476 ft.: find the length of the third side.

2. The three different edges of a rectangular parallelopiped are 3 ft., 2·52 ft., and 1·523 ft.: find the cubical contents of the box, if the material of which it is composed is $\frac{1}{10}$ of a foot in thickness.

3. An hexagonal pyramid whose height is 8 ft. has each of its sides at the base 6 ft.: how far from the top must a plane be drawn, parallel to the base, to divide the pyramid into two equal parts?

4. What is the cost of papering a square room 22 ft. 6 in. in length and 12 ft. 4 in. in height, with paper 2 ft. 4 in. wide, at 9*d.* per yard?

5. If the number of square feet on the surface of a sphere equals the number of cubic feet in its volume, what number represents its diameter?

6. Find the cost of plating a cube of metal containing 2 cub. ft. 1457 cub. in. with silver at 3*s.* 9*d.* per square foot.

VI.

1. The perimeter of a square is 1 ft.: compare its area with that of an equilateral triangle whose perimeter is 1 ft.

2. The sides of three cubes have equal differences, their sum is 15 in., and the solid contents of the three cubes together are 495 cub. in.: required the length of the side of each cube.

3. How many marbles, each one inch in diameter, can be packed in a box whose internal dimensions are an exact cubic foot?

4. Two sides of a triangular field containing an obtuse angle are 110 yds. and 220 yds. respectively: find the length of the third side that the field may contain exactly an acre.

5. How many square feet of metal will be required to make a rectangular tank, open at the top, 12 ft. long, 10 ft. broad, and 8 ft. deep?

6. A circular pond has an area of $346\frac{1}{2}$ sq. yds.: find to the nearest penny the cost of fencing it round at 4*s.* 6*d.* per yard.

VII.

1. The sides of a triangular field are 10 ch., 8 ch., and 12 ch.: find the acreage of the field.

2. A cubical cistern contains when full 2000 cub. ft. of water: find the length of one of its sides to the tenth of an inch.

3. What quantity of metal will be required to make a hollow spherical ball, the external diameter being 18 in. and the thickness $4\frac{1}{2}$ in.?

4. Find the area of a quadrilateral figure in which one of the diagonals is 93 ft., and the perpendiculars upon it from the opposite angles are 60·5 ft. and 36·7 ft. respectively.

5. The dimensions of a rectangular box are as 2 : 3 : 4, and the difference between the cost of covering it with sheet lead at 8*d.* and $8\frac{1}{2}$*d.* is 4*s.* $10\frac{1}{2}$*d.*: what are the dimensions of the box?

6. What is the area of the largest circle that can be inscribed in the square whose area is 5499025 sq. ft.?

VIII.

1. If the side of a cube be 2, required the side of a cube exactly double the contents of the former.

2. Find the number of cubic feet in a room which is 36 ft. long, 20 ft. wide, and 10 ft. high to the spring of the roof, and 16 ft. from the floor to the pitch of the roof.

3. Find the weight of a ball of ash wood whose specific gravity is $\frac{4}{5}$ of water, the diameter of the ball being 10 in., and the weight of a cubic foot of water 1000 ozs.

4. The sum of £9 0*s.* 10*d.* is allowed for papering a room 27·7 ft. long, 19·55 ft. wide, and 12·4 ft. high: how much per yard must be given for a paper 2·7 ft. wide?

5. Find the area of the six equal faces of an hexagonal pyramid, each side of the base being 6 ft., and the perpendicular height 8 ft.

6. Find the surface of a globe whose diameter is 24 in.

IX.

1. If the radius of a given circle is 1 ft., find to the hundredth of an inch the radii of the two concentric circles which divide the area into three equal parts.

2. If three yards be taken from one side of a rectangle whose perimeter is 14 yds., and added to the other side, its area will be doubled: find the length of the sides.

3. How many superficial feet of 1 in. plank can be sawn out of a log of timber 20 ft. 7 in. long, 1 ft. 10 in. broad, and 1 ft. 8 in. deep?

4. Find the weight of a bombshell whose exterior and interior diameters are 10 in. and 8 in. respectively. Let $\pi = \frac{22}{7}$, and the specific gravity of the iron be 7·21 times that of water.

5. The walls of a room 21 ft. long, 15 ft. 9 in. wide, and 11 ft. 8 in. high, are painted for £17 17*s.* $3\frac{1}{2}$*d.*: find the expense of painting the ceiling at the same rate.

6. A cylindrical vat 8 ft. in diameter contains 960 gals.: what is its depth, supposing a gallon equivalent to 275 cub. in.?

X.

1. Find to the nearest hundredth of an inch the radius of a sphere whose surface is 1 sq. ft.

2. If a room 40 ft. long by 20 ft. broad contains 12800 cub. ft., what addition will be made to the cubic contents by throwing out a semi-circular bow at one end?

3. How many square feet of canvas are necessary for a conical tent whose perpendicular height is 10 ft., and diameter of the base 13 ft.?

4. A reservoir containing 1200 cub. ft. of water is emptied by means of a pipe 3 in. in diameter: if the water flows out with a velocity of 2 miles an hour, how long will it take to empty it?

5. A cubical box exactly holds 125 shot, each 3 in. in diameter: find how many cubic inches of sand would be required to fill up the interstices.

6. Find to the nearest square inch the quantity of leather required to cover a spherical ball 23 in. in circumference.

XLIV.—EASY EXERCISES.

These exercises are adapted either for mental or rapid book-work for the purposes of revision.

I.—On Measurements.

A.

1. What is the difference between 5 sq. ft. and 5 ft. square?
2. What is the difference between 3 cub. ft. and 3 ft. cube?
3. How many yards in 5 ch.?
4. How many chains in $4\frac{1}{2}$ miles?
5. How many square yards in 1 sq. ch.?
6. How many square chains in 1 ac.?

B.

1. Reduce 450000 sq. lks. to acres.
2. How many acres in 7595 sq. ch.?
3. How many acres in 375000 sq. lks.?
4. How many feet in 3 ch.?
5. In 1 mile 5 ch., how many yards?
6. How many chains measure a furlong?

C.

1. Multiply $17\frac{1}{2}$ ft. by $3\frac{3}{7}$ ft.
2. Multiply 49 ft. by 12·5 ft.
3. Multiply 25 yds. by 33 yds.
4. Divide 795 sq. ft. by 15 ft.
5. Multiply $10\frac{1}{2}$ ft. by $7\frac{1}{3}$ ft.
6. Multiply $8\frac{3}{4}$ ft. by $8\frac{4}{7}$ ft.

II.—On Parallelograms.

A.

1. How do you find the area of a parallelogram?

2. Find the area of a floor in square yards which measures 15 ft. by 12 ft.

3. How many square inches in a table 3 ft. 6 in. long and 2 ft. 1 in. broad?

4. How many acres in a field which measures 10·5 ch. by 12·5 ch.?

5. How many yards of carpet 1 yd. wide will cover a room 18 ft. by 15 ft. in measurement?

6. What is the area of a rectangular playground which measures 45 yds. by 25 yds.?

B.

1. How do you find the area of a square?

2. How do you find the area of a square when the diagonal is given?

3. Find the area of a square field whose side is 500 lks.

4. Find the area of a square table whose diagonal is 12 ft.

5. A square playground is 20 yds. long: what will it cost to asphalt at 8*d.* a square yard?

6. What length of rope in yards will surround a rectangular field which measures $4\frac{1}{2}$ ch. by $3\frac{1}{2}$ ch.?

C.

1. Given the area of a square, how is the length of the side found?

2. Given the area of a square, how is the length of the diagonal found?

3. The area of a square garden is one-tenth of an acre, find the length of the side in yards.

4. The area of a square cricket-field is $2\frac{1}{2}$ ac., find the length of the side in yards.

5. The area of a square field is 5 ac., find its diagonal.

6. A room is 15 ft. square: what will it cost to carpet at 3*s*. 6*d*. a square yard?

D.

1. Given the area of a parallelogram and one side, how is the perpendicular height found?

2. The area of a plank is 27 sq. ft., its breadth is 9 in.: find its length.

3. A black-board of 2 sq. yds. is $4\frac{1}{2}$ ft. long: find its breadth.

4. A room whose area is 30 sq. yds. is 15 ft. broad: find its length.

5. What will it cost to carpet a room 12 ft. by 18 ft. at 4*s*. a square yard?

6. A wall is 18 ft. long and 15 ft. high: find the cost of painting it at 3*d*. a square yard.

III.—On Triangles.

A.

1. Given the base and perpendicular of a right-angled triangle, to find the hypothenuse.

2. Find the length of a ladder which will reach 12 ft. high 9 ft. from the wall.

3. How high will a ladder 25 ft. long reach placed 15 ft. from the wall?

4. A ship sails east 30 miles, another ship north for 40 miles: how many miles are they then apart if they sailed from the same port?

5. The area of a square field is 10 acres: what is the length of the rope in yards which will surround it?

6. The diagonal of a square is 15 ft.: find the area.

B.

1. Given the base and perpendicular of a triangle, to find the area.

2. Find the area of a triangular field whose base is 5 ch., and perpendicular height 1 ch.

3. Find the area of a triangle whose base is 15 ft., and perpendicular height 10 ft.

4. Find the cost of gravelling a triangular courtyard 15 ft. long, and whose perpendicular breadth is 8 ft., at 5*d*. a square yard.

5. How many stones 2 ft. by $1\frac{1}{2}$ ft. will pave a courtyard 17 ft. by 21 ft.?

6. What will it cost to paper a wall 16 ft. by 12 ft., at 3*d*. a square yard?

C.

1. What is the area of a triangle whose three sides are a, b, and c?

2. Find the area of a triangle whose sides are 4 ft., 8 ft., and 10 ft.

3. Find the area of a triangle whose sides are 4 in., 6 in., and 8 in.

4. The area of a triangular slab is 90 sq. ft.; its perpendicular height is 12 ft.: find its base.

5. The length of a rectangle is 9 ft., its breadth 12 ft.: find the length of the diagonal.

6. The base and perpendicular of a right-angled triangle are 6 ft. and 8 ft.: find the length of the perpendicular from the right angle on the hypothenuse.

IV.—Miscellaneous.

A.

1. Reduce 725000 sq. lks. to acres.

2. Multiply $10\frac{1}{2}$ ft. by $11\frac{1}{3}$ ft.

3. In 2 miles 3 ch. how many yards?

4. Find the area of the walls of a room which measures 12 ft. long, 10 ft. broad, and 9 ft. 6 in. high.

5. How many yards of carpet 2 ft. 3 in. wide will cover a floor 12 ft. long by $11\frac{1}{4}$ ft. wide?

6. The hypothenuse of a right-angled triangle is 12 ft., and the area is 48 sq. ft.: find the length of the perpendicular on the hypothenuse from the right angle.

B.

1. A quadrilateral has two parallel sides 18 ft. and 24 ft., and perpendicular distance between them 14 ft.: find the area.

2. Find the area of a trapezoid whose parallel sides are 5 ft. and 8 ft., and perpendicular distance between them 4 ft.

3. A board has two parallel sides of 1 ft. 10 in. and 1 ft.; the perpendicular width is 10 in.: find the area.

4. The area of a square field is 1·6 ac.: find its perimeter in yards.

5. What is the side of a square room equal in area to a room 25 ft. by 9 ft.?

6. The area of a triangle is 84 sq. yds., and the perpendicular on the base 42 ft.: find the length of the base.

V.—On Circles.

A.

1. How do you find circumference, diameter given?
2. What does π represent?
3. If the radius of a circle is $3\frac{1}{2}$ ft., find the circumference.
4. If the diameter of a circle is 7 ft., find the circumference.
5. If the radius of a wheel is $1\frac{3}{4}$ ft., find its circumference.
6. A square field is 1 ac. 1 ro.: find the radius of the circumscribing circle.

B.

1. How is the area of a circle found from the diameter?
2. How is the area of a circle found from the radius?
3. Find the area of a circle whose diameter is $3\frac{1}{2}$ ft.
4. Find the area of a circle whose diameter is 7 ft.
5. Find the area of a circle whose radius is 2 ft.
6. Find the circumference of a circle whose radius is 14 ft.

C.

1. How is the diameter of a circle found from the circumference?
2. How is the radius of a circle found from the circumference?
3. Find the diameter of a circle whose circumference is $9\frac{3}{7}$ ft.
4. Find the radius of a circle whose circumference is 66 ft.
5. Find the circumference of a wheel whose radius is $2\frac{1}{3}$ ft.
6. Find the diameter of a circular cricket-ground which is one mile in circumference.

D.

1. How do you find the area of a circle from the circumference?
2. How many acres are there in a circle whose circumference is one mile?
3. What is the area of a ring 22 yds. in circumference?
4. Compare the area of a circle whose diameter is 4 with a square whose side is 4.
5. Compare the circumference of a circle whose diameter is 7 with the perimeter of a square whose side is 7.
6. What is the area of a circular plot whose radius is 5 ft.?

E.

1. How do you find the diameter of a circle from the area?
2. How do you find the radius of a circle from the area?

3. What is the diameter of a circle whose area is 154 sq. yds.?

4. What is the radius of a circle whose area is $12\frac{4}{7}$ sq. ft.?

5. What is the diameter of a circle in feet whose area is $3\frac{1}{7}$ sq. yds.?

6. Compare the diameter of a circle with its circumference.

F.

1. How do you find the circumference of a circle from the area?

2. Find the circumference of a circle whose area is 154 sq. yds.

3. Find the circumference of a circle whose area is $38\frac{1}{2}$ sq. yds.

4. Find the circumference of a circle whose area is $9\frac{5}{8}$ sq. ft.

5. Compare the diagonal of a square whose side is 7 ft. with the diameter of a circle of $10\frac{1}{2}$ ft. radius.

6. What is the side of a square field containing 10 ac.?

G.

1. How do you find the area of a circular ring?

2. What is the area of a ring whose outer and inner diameters are 22 in. and 8 in.?

3. Find the area of a ring whose outer and inner diameters are 28 ft. and 14 ft.

4. Find the area of a road 1 ch. broad round a circular plantation whose diameter is 24 ch.

5. Find the diagonal of a square whose area is 50 sq. ft.

6. The radius of a carriage-wheel is $1\frac{3}{4}$ ft.: how many times will it turn round in travelling 2 miles?

VI.—On Sectors.

A.

1. What is a sector of a circle?

2. How do you find the area of a sector from the length of the arc and radius of the circle?

3. Find the area of a sector whose radius is 5 ft., and length of arc 14 ft.

4. Find the area of a sector the length of whose arc is 14 yds., and radius of the circle $2\frac{1}{2}$ yds.

5. Find the area of a sector whose arc is 16 ft. long, and the radius of the circle $12\frac{1}{2}$ ft.

6. The radius of a quadrant is 7 ft.: find the area.

B.

1. What is the area of a sector of a circle whose arc contains 126°, and the radius of the circle 5 ft.?

2. What is the area of the sector of a circle whose arc contains 140°, and the radius of the circle 6 ft.?

3. What number of degrees is there in a sector of a circle whose radius is 7 ft., and length of the arc 11 ft.?

4. What number of degrees is there in a sector of a circle whose radius is $1\frac{3}{4}$ ft., and length of arc $3\frac{2}{3}$ ft.?

5. What is the area of a circle a sector of which, having an angle of 36°, contains 12 sq. ft.?

6. Find the area of a semicircle whose radius is $10\frac{1}{2}$ ft.

VII.—On Chords and Segments.

1. What is a chord?

2. What is a segment?

3. What is the length of the chord of a circle 3 in. from the centre, the radius being 5 ft.?

4. What is the radius of a circle in which a chord of 6 ft. is 4 ft. from the centre?

5. What is the radius of a circle of which a chord is 8 ft., and the height of the arc 2 ft.?

6. What is the radius of a circle when a chord of 8 ft. has the height of the arc 4 ft.?

VIII.—On the Ellipse.

1. How do you find the area of an ellipse?

2. Find the area of an ellipse whose diameters are 10 ft. and 7 ft.

3. Find the area of an ellipse whose semidiameters are 7 ft. and 6 ft.

4. Find the area of a field in the form of an ellipse whose diameters are 10 ch. and 14 ch.

5. Find the area of an ellipse whose semidiameters are $10\frac{1}{2}$ ft. and 8 ft.

6. Compare the area of a circle whose radius is 7 ft. with an ellipse whose semidiameters are 7 ft. and 6 ft.

IX.—Miscellaneous.

1. Find the area of a square field whose diagonal is 16 ch.

2. The area of a square table is 4 sq. ft. 49 sq. in.: find the length of the side.

3. Find the cost of carpeting a room 18 ft. by 24 ft. at 2*s*. 6*d*. a square yard.

4. What is the diameter of a circle whose circumference is 33 ft.?

5. Find the side of a square equal in area to a rectangle 18 ft. by 8 ft.

6. What is the area of a square field whose diagonal is 10 ch.?

X.—On Standards of Capacity.

1. What is the weight of a gallon of water?
2. What is the cubic measurement of a gallon?
3. What is the weight of a cubic foot of water?
4. How many gallons make a cubic foot of water?
5. What is the weight of a pint of water?
6. What are the cubic contents of a pint of water?

XI.—On the Parallelopiped.

A.

1. What are the solid contents of a 4-ft. cube?
2. What is the whole surface of a 4-ft. cube?
3. What weight of water will a 2-ft. cubical cistern hold?
4. What are the contents of a box 3 ft. by 2 ft. by $1\frac{1}{2}$ ft.?
5. Find the whole surface of a box 3 ft. by 2 ft. by $1\frac{1}{2}$ ft.
6. The surface of a cube is 216 sq. yds.: find the length of one side.

B.

1. A tank measuring 1680 cub. yds. is 20 yds. long and 14 yds. broad: find its depth.

2. How many cubic feet of air are there in a room 20 ft. by 12 ft. and 10 ft. high?

3. What size cube contains 1000 cub. in.?

4. A log of wood of 60 cub. ft. is 20 yds. long, $\frac{3}{4}$ ft. wide: find its thickness.

5. How many gallons will a cistern 5 ft. by $2\frac{1}{2}$ ft. by 2 ft. deep hold?

6. What is the weight of water in a cistern 3 ft. by 2 ft. and 2 ft. deep, when it is full?

C.

1. How many pieces of wood, each 3 in. by 2 in. by 4 in. thick, can be cut from a cubic foot?

2. What is the cost of 3 pieces of wood, each 16 ft. by 1 ft. and $1\frac{1}{2}$ in. thick, at 2*s.* 6*d.* per cubic foot?

3. What weight of water in tons will a reservoir hold which is 16 yds. by 14 yds. and 10 ft. deep?

4. How many blocks, each 3 in. by 2 in. and 2 in. deep, will fill a box $1\frac{1}{2}$ ft. by 1 ft. and 6 in. deep?

5. What is the total surface of a box which is 3 ft. by 2 ft. by $2\frac{1}{2}$ ft. deep?

6. What must be the depth of a cistern 5 ft. by 4 ft., to hold 250 gals. of water?

XII.—On Prisms and Cylinders.

1. How do you find the volume of a prism?

2. Find the volume of a triangular prism whose sides at base are 3 in., 4 in., and 5 in., and perpendicular height 2 ft.

3. Find the volume of a cylinder whose diameter is $3\frac{1}{2}$ ft., and height 10 ft.

4. Find the cost of sinking a well 7 ft. in diameter, 33 ft. deep, at 7 cub. ft. for 6*d.*

5. How many gallons are there in a well $3\frac{1}{2}$ ft. in diameter, having a depth of 20 ft. of water?

6. Find the volume of a prism whose area at the base is 5 sq. ft., and height 2 ft. 6 in.

XIII.—On Cones and Pyramids.

1. How do you find the solidity of a cone?

2. Compare the volumes of a cone and cylinder having the same base and perpendicular height.

3. How do you find the slant height of a cone, having given the perpendicular height and radius of the base?

4. How do you find the upright surface of a cone?

5. Find the volume of a cone whose base is 100 sq. in., and whose height is $1\frac{1}{2}$ ft.

6. Find the volume of a square pyramid whose side is 4 ft., and height 12 ft.

XIV.—Miscellaneous.

1. Find the slant height of a cone the diameter of whose base is 30 ft., and perpendicular height 20 ft.

2. Find the upright surface of a cone the circumference of whose base is 27 ft., and slant height 50 ft.

3. What is the circular surface of a cylinder whose diameter is 7 ft., and perpendicular height 20 ft.?

4. What is the volume of a cone, the diameter of the base being 7 ft., and height 30 ft.?

5. Find the volume of a triangular pyramid the sides of whose base measure 3 in., 4 in., and 5 in., and whose height is 15 in.

6. How many tons of water will a cylinder hold whose diameter is 7 ft., and height 20 ft.?

XV.—On the Sphere.

1. How do you find the volume of a sphere?

2. Compare the volume of a cube with that of a sphere whose diameter equals the side of the cube.

3. Compare the volumes of a sphere and its circumscribing cylinder.

4. What are the contents of a sphere whose diameter is 3½ ft.?

5. Find the contents of a sphere whose radius is 1 ft.

6. Compare a cube of 1 ft. and a sphere whose diameter is 1 ft.

XVI.—Miscellaneous.

A.

1. Compare the areas of two similar triangles whose bases are 3 in. and 4 in. respectively.

2. Compare the volumes of two spheres whose radii are respectively 3 in. and 4 in.

3. Compare the areas of two squares whose sides are respectively 3 in. and 4 in.

4. Compare the volumes of two cubes whose sides are respectively 3 in. and 4 in.

5. Compare the volume of a cube whose diagonal is 3 with a sphere whose diameter is 3.

6. Compare the volumes of two similar solids whose perpendiculars are 2 ft. and 3 ft. respectively.

B.

1. A triangle has a base of 50 in.: find the length of a line parallel to the base which divides the triangle equally.

2. A triangle has a base 50: find the lengths of two lines parallel to the base which divide the triangle into three equal parts.

3. Find the surface of a cube whose side is 3 ft.

4. Find the surface of a sphere whose diameter is 3 ft.

5. Find the canvas required to make a conical tent 8 ft. high, whose diameter at the base is 12 ft.

6. How far can you see at a height of 5 miles, supposing the diameter of the earth = 8000 miles?

C.

1. How do you find the area of a rhombus?

2. Find the area of a rhombus whose diagonals are 7 ft. and 10 ft.

3. Find the area of an equilateral triangle whose side is 5 ft.

4. Find the area of an hexagon whose side is 1 ft.

5. Compare the areas of two similar regular polygons whose sides are 4 ft. and 5 ft. respectively.

6. The area of a polygon whose side is 5 ft. is 85 sq. ft.: find the area of a similar polygon whose side is 3 ft.

D.

1. If 1 sq. in. represents a square yard, what is the scale of the plan?

2. If 1 sq. ft. represents 10 ac., what is the scale?

3. The sides of a rectangle are as 2 : 3, and the area is 150: find the sides.

4. A field of 1 ac. is represented on a plan by 1 sq. in.: what is the scale?

5. Compare the areas of a square and hexagon whose perimeters are equal.

6. How many plots, each 22 yds. square, can be made from 3 ac.?

E.

1. What is the side of a square whose area is 6·25 sq. ft.?

2. Find the area of a square whose diagonal is 5 ft.

3. Find the area of the walls of a room 13 ft. 6 in. by 12 ft. 6 in., and 10 ft. high.

4. Compare the perimeters of a square and equilateral triangle of equal areas.

5. How many square yards of carpet will cover a room 15 ft. by 12 ft.?

6. What is the area of a circle whose radius is $2\frac{1}{2}$ ft.?

F.

1. What is the volume of a cylinder 7 ft. radius and 10 ft. high?

2. What is the diameter of a circle whose area is 154 sq. yds.?

3. Find the area of a circular ring whose inner and outer diameters are 8 ft. and 6 ft.

4. An elliptical flower-bed has its conjugate diameters 14 ft. and 10 ft.: find its area.

5. What is the surface of a sphere whose radius is 7 ft.?

6. What is the total surface of a cone whose radius is 3 and slant height 11?

G.

1. Compare the volumes of two globes, one having a diameter of 2 ft., the other a circumference of 2 ft.

2. Find the cost of painting the ceiling of a room 15 ft. by 18 ft., at 2*s.* a square yard.

3. Find the slant height of a cone whose perpendicular height is 12 ft., and diameter of the base 18 ft.

4. Compare the surfaces of a cube whose side is 1 ft. and of a sphere whose diameter is 1 ft.

5. Find the volume of a square pyramid whose base is 3 ft. square, and height 10 ft.

6. A map is drawn $\frac{1}{16}$ of an inch to a mile: how far apart are two places which on the map are $3\frac{1}{2}$ in. apart?

H.

1. Find the area of the walls of a room 14 ft. by 16 ft., and 12 ft. high.

2. A square field has a side of 100 yds.: how far is it from one corner to the opposite one?

3. Find the lateral surface of a cone whose radius is $10\frac{1}{2}$ ft., and slant height 14 ft.

4. Find the area of a circular cricket-ground whose diameter is 21 ch.

5. How many plots, each 11 yds. square, are there in $1\frac{1}{2}$ ac. of ground?

6. How many times will a wheel whose radius is $1\frac{3}{4}$ ft. turn round in travelling one mile?

I.

If a room is 15 ft. long, 14 ft. wide, and 10 ft. high, find—

1. Area of the floor in square yards.
2. Length of the walls in feet.
3. Area of the walls in square yards.
4. Cost of covering the floor with carpet a yard wide, at 3*s*. 6*d*. a yard.
5. Cost of colouring the walls and ceiling, at 4*d*. a square yard.
6. Cost of covering the floor with carpet 2 ft. 3 in. wide, at 4*s*. 6*d*. a yard.

K.

1. Find the surface of a globe whose radius is $3\frac{1}{2}$ ft.

2. Find the volume of a sphere whose diameter is $1\frac{3}{4}$ ft.

3. What is the surface of the segment of a sphere whose radius is 7 ft., and height 3 ft.?

4. What will it cost to gild the surface of a ball 7 in. in diameter at 1*s*. 6*d*. a square inch?

5. Find the volume of a spherical segment of a sphere whose radius is $3\frac{1}{2}$ ft., and height $1\frac{1}{2}$ ft.

6. What must be the diameter of a sphere that its surface may be 616 sq. ft.?

XLV.—ON THE RATIO OF THE AREA OF A REGULAR POLYGON TO THE SQUARE ON ITS SIDE.

In Section X. (*a*), p. 24, the area of a polygon is found from the side and the radius of the inscribed circle.

Without knowing the radius of the inscribed circle, the area of the polygon may be found from the length of one of the sides.

It has been shown (Sect. X. (*b*)) that the area of a hexagon equals the square of the length of one of its sides multiplied by 2·598. Also, in Sect. VIII. (*e*), that the area of an equilateral triangle equals the square of the length of the side multiplied by ·433.

It can be shown that the area of any polygon bears a constant ratio to the square on one of its sides.

The following table gives the ratio of the area of the polygons named to the square on one of its sides :—

Number of sides.	Name.	Ratio of area to square on the side.
3	Equilateral triangle	·433
4	Square	1·
5	Pentagon	2·72
6	Hexagon	2·598
7	Heptagon	3·634
8	Octagon	4·828
9	Nonagon	6·182
10	Decagon	7·694
11	Undecagon	9·366
12	Duodecagon	11·197

Thus, if the length of a side of any given polygon be given, the area of the polygon can be found by multiplying the square of the length of the side by the given ratio for that polygon in the above table.

Example.—Find the area of a regular nonagon whose side is 5 in.

$$\text{Area} = 5^2 \times 6{\cdot}182$$
$$= 25 \times 6{\cdot}182 = 154{\cdot}5 \text{ sq. in.} \quad \text{Ans.}$$

XLVI.—ON THE RATIO OF THE RADIUS OF THE INSCRIBED CIRCLE IN A POLYGON TO THE SIDE OF THE POLYGON.

It has been shown in Section VIII. (*e*) that the perpendicular from the apex to the opposite side of an equilateral triangle equals the length of the side multiplied by $\frac{\sqrt{3}}{2}$. Therefore the radius of the inscribed circle in a hexagon equals the length of the side of the hexagon multiplied by ·866.

It can be shown that the radius of the inscribed circle in a polygon bears a constant ratio to the side of the polygon.

The following table shows the ratio of the inscribed circle to the side of the polygon:—

Number of sides.	Name.	Ratio of radius of inscribed circle to side of polygon.
3	Equilateral triangle	·2887
4	Square	·5
5	Pentagon	·6882
6	Hexagon	·866
7	Heptagon	1·0383
8	Octagon	1·2071
9	Nonagon	1·3737
10	Decagon	1·5388
11	Undecagon	1·7028
12	Duodecagon	1·886

Thus if the length of the side of a polygon be given, the area of the polygon can be found by Section X. (*a*), for the radius of the inscribed circle will be the length of the side of the polygon multiplied by the given ratio.

Example.—Find the area of a regular nonagon whose side is 5 in.

Length of radius of inscribed circle $= 5 \times 1·3737$

$$\text{Area of nonagon} = \frac{5 \times 9 \times 5 \times 1·3737}{2}$$

$$= 154·5 \text{ sq. in.} \quad \text{Ans.}$$

XLVII.—TABLE OF FORMULÆ.

1. **Right-angled triangle.** $h = \sqrt{p^2 + b^2}$.

2. **Square.** (i.) $A = s^2$; (ii.) $A = \frac{d^2}{2}$.

3. **Rectangle.** $A = l \times b$.

4. **Rhombus.** $A = \frac{d_1 \times d_2}{2}$, where d_1 and d_2 are the diagonals.

5. **Triangle.** (i.) $A = \frac{b \times h}{2}$; (ii.) $A = \sqrt{s \cdot \overline{s-a} \cdot \overline{s-b} \cdot \overline{s-c}}$, where a, b, c are the three sides, and $s = \frac{a+b+c}{2}$.

6. **Equilateral triangle.** $A = \frac{s^2}{4}\sqrt{3}$.

7. **Polygon.** $A = \frac{n \times s \times r}{2}$, where n = number of sides, s = length of side, and r = radius of the inscribed circle.

8. **Regular hexagon.** $A = \frac{3s^2\sqrt{3}}{2}$.

9. **Similar rectilineal figures.** (i.) Corresponding sides are proportional. (ii.) Areas are as the squares of corresponding sides.

10. **Chords of circles.** Chord of whole arc $= 2\sqrt{h(d-h)}$, where h = height of segment, and d = diameter of circle; chord of half the arc $= \sqrt{2rh}$.

11. **Circle.** Circumference $= 2\pi r$; $A = \pi r^2$.

12. **Circular rings.** $A = \pi(R + r)(R - r)$, R and r being the external and internal diameters respectively.

13. **Sectors.** $A = \frac{lr}{2}$, where l = length of arc, and r = radius of circle.

14. **Ellipse.** $A = \pi r r_1$, where r and r_1 are the semi-diameters.

15. **Cube.** $V = s^3$; $S = 6s^2$; Diagonal $= s\sqrt{3}$.

16. **Parallelopiped.** $V = lbh$; $S = 2(lb + lh + bh)$.

17. **Prism.** $V = A \times h$, where A = area of base.

18. **Cylinder.** $V = \pi r^2 h$; lateral $S = 2\pi rh$; total $S = 2\pi r(h + r)$.

19. **Circular rings.** $V = \frac{1}{32}(C - c)^2 (D + d)$.

20. **Flat ring.** $V = \frac{\pi h}{4}(D^2 - d^2)$.

21. Pyramid. Lateral $S = \frac{\text{perim.} \times \text{slant } h}{2}$; $V = \frac{\text{area of base} \times h}{3}$

22. Cone. Lateral $S = \pi r h$; total $S = \pi r(r + h)$, where h = slant height; $V = \frac{\pi r^2 h}{3}$, where h = perpendicular height.

23. Frusta of pyramid. Lateral S = (perimeter of base + perimeter of top) $\times \frac{\text{slant } h}{2}$; $V = \frac{h}{3}(S_1 + \sqrt{S_1 S_2} + S_2)$, where h = perpendicular height, and S_1 and S_2 areas of base and top respectively.

24. Frusta of cones. Lateral $S = \pi h(R + r)$; total $S = \pi\{R^2 + r^2 + h(R + r)\}$, where h = slant height; $V = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$, where h = perpendicular height.

25. Cask. $V = \frac{\pi h}{6}(3R^2 + Rr + 2r^2)$, where h = height, R = bung radius, and r = end radius.

26. Wedge. $V = \frac{dh}{6}(e + 2b)$, where d = breadth of base, b = length of base, e = edge, and h = height.

27. Prismoid. $V = \frac{h}{6}(4A + db + mn)$, where A = area of mean section, db = area of base, mn = area of top.

28. Similar rectilineal solids. (i.) Volumes are as cubes of corresponding sides. (ii.) Volumes of similar rectilineal solids are as the products of their dimensions.

29. Sphere. $S = 4\pi r^2$; $V = \frac{4\pi r^3}{3}$.

30. Zone. $S = c \times h = 2\pi r h$, where h = height of zone; $V = \frac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$, where r_1 and r_2 = radii of two ends.

31. Spherical shell. $V = \frac{\pi}{3}\left(\frac{D - d}{2}\right)(D^2 + Dd + d^2)$, where D and d are diameters of the outer and inner circumferences of shell.

32. Spherical sector. $S = 2\pi r h$; $V = \frac{2\pi r^2 h}{3}$.

33. Spherical segment. $S = 2\pi r h$. V = (i.) $\pi h^2\left(r - \frac{h}{3}\right)$; (ii.) $\frac{\pi h^2}{6}(3D - 2h)$; (iii.) $\frac{\pi h}{6}(3b^2 + h^2)$, where b = radius of base.

XLVIII.—METHODS OF SOLUTION.

In the solution of the more difficult problems in mensuration, time and trouble may be saved by a judicious choice of methods. The following suggestions are offered as an assistance to students:—

I. Draw a figure, in order to get a clear grasp of the meaning of the question.

II. Obtain the final result by using multiplying fractions without any intermediate reasoning on paper.

III. Do not commence the arithmetical calculation until the final result has been obtained as a complex fraction.

IV. Do not insert the value of π until this final stage is reached.

V. Use contracted methods in multiplication and division, square and cube root.

VI. Use logarithm tables whenever time may be saved in doing so.

VII. Do not confuse *radius* and *diameter* of a circle, *slant height* and *height*, *volume* and *surface*, etc.

VIII. Employ the same unit of measurement throughout the calculations.

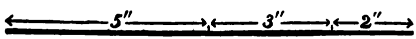
IX. Prefer decimals to ordinary fractions, and short division to long.

X. Discard all small quantities which do not affect the required accuracy of your answer.

XI. Apply rough tests to see if your answer is approximately correct; and see if the answer is sensible.

Some remarks and examples are added to illustrate these rules.

I. In denoting lengths, the abbreviations, such as 2′, 3″ (two feet, three inches), will be found useful. When a portion of a line is denoted, the length may be specified as follows:—



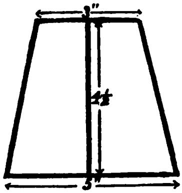
II. The method of using multiplying fractions is illustrated in this example:—

Find the contents of a mug in the shape of a frustum of a cone, 4½ in.

high, with the diameter of the top and base 3 in. and 5 in.: given that 1 gal. of water weighs 10 lbs., and a cubic foot of water weighs 1000 ozs.

$$\text{The volume of the mug} = \frac{1}{3} \times \frac{9}{2}\left\{\pi\left(\frac{3}{2}\right)^2 + \pi\frac{3}{2} \times \frac{5}{2} + \pi\left(\frac{5}{2}\right)^2\right\}$$
$$- \frac{3\pi}{2} \times \frac{49}{4} \text{ cub. in.}$$

$$\text{hence it contains } \frac{3 \times 22 \times 49 \times 1000 \times 8}{2 \times 7 \times 4 \times 1728 \times 16 \times 10} \text{ pints}$$
$$= 1{\cdot}671 \text{ pints.}$$



For, after cancelling,

8)7700
12)962·5
12)80·208
4)6·684
1·671

In this example the volume is divided by 1728 to obtain cubic feet, multiplied by 1000 to obtain ounces, divided by 16 to obtain pounds, divided by 10 to obtain gallons, multiplied by 8 to obtain pints.

III. The obvious advantage of this method consists in the increased chance of numbers cancelling; and it is also easy to check the accuracy of the reasoning. The above example serves as an illustration.

IV. The value inserted for π is necessarily approximate, and there is increased accuracy when π is cancelled without substitution for its value. Of course, this can only be done occasionally, but in all cases there is trouble saved by inserting the value of π in the final stage.

Example.—Five hundred spherical bullets. $\frac{1}{2}$ in. in diameter, are recast as a cone whose height is equal to the diameter of the base: find the height of the cone.

Let x equal the height of the cone.

$$\therefore \frac{x}{3} \cdot \pi \cdot \frac{x^2}{4} = 500 \times \frac{4}{3}\pi\left(\frac{1}{4}\right)^3$$
$$\therefore x^3 = 125$$
$$x = 5 \text{ in.}$$

V. Contracted methods are explained in all good text-books on arithmetic, but elaborate details are not required in applying these rules.

In multiplication, work from the *left* of the multiplier, and move successive rows to the right. Use your own judgment as to where contraction may begin, then cut off figures of the multiplicand from the right, and keep the right-hand figures of the rows in a straight line. Always carry forward the figure derived from the digit last cut off. The position of the decimal point can be readily determined by considering what are the approximate numbers which are being multiplied together.

Example.—Multiply 320·89756 by 35·342 to three places of decimals.

```
  320·89756
   35·342
  ---------
  96269268
  16044878
    962692
    128359
      6418
  ---------
 11341·1615
```

A margin of safety has been allowed, but the last figure cannot be relied upon with certainty. The decimal point was inserted by noticing that 300 was to be multiplied by 35, roughly speaking; so that the answer should be rather above 10,500.

In division, the last figures of the divisor are cut off in succession, taking care that sufficient accuracy is obtained. The figure obtained from the last digit cut off must be brought forward in the various products of the divisor.

Example.—Divide 11341·1615 by 35·342 to four places of decimals.

```
35342)1134l161·5 (320·89756
      106026
      ------
       73856
       70684
       ------
        317215
        282736
        ------
         34479
         31808
         -----
          2671
          2474
          ----
           197
           176
           ---
            21
```

Square root.—Consider the total number of digits required in the answer, add one or two to this number, and divide by 2. When the number of digits in the quotient is equal to this result, the remaining digit can be obtained by ordinary contracted division. In short, after the square root is half finished, the work is completed by division.

Example.—Find the square root of 1187·562058 to six places of decimals. In this case eight figures are required in the quotient, and contracted division can be used after proceeding to five figures in the quotient.

```
        1187·562058 (34·4610223
         9
       ----
    64) 287
        256
       ----
   684) 3156
        2736
        ----
  6886) 42020
        41316
        -----
  68921) 70458
         68921
         -----
          1537
          1378
          ----
           159
           138
           ---
            21
```

Cube root.—It is best to use logarithms whenever tables are at hand; otherwise Horner's method is the best. A fairly rapid and easy rule may be used in mensuration, which is deduced from the algebraical formula—

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

After three or four figures have been obtained in the quotient, the next two or three digits can be obtained by ordinary division. A brief summary of the rule is given for reference.

Divide the digits into groups of threes from the right and left of the decimal point. Find the number whose cube is the nearest below the first group, and place it in the quotient. Cube and subtract from the first group, and bring down the second group. Take the quantity in the quotient, multiply it by 30, and put in a column on the left. Square the quantity in the quotient, multiply it by 300, and place it in a middle column. The latter is called the trial divisor; and if we suppose that it goes four times, the 4 is written under the first column, and added to the line just above. Multiply the result by 4, and place it under the middle column; add, and again multiply by 4, and place it under the main column. Subtract, and repeat the whole process with the total quotient. As the trial divisor is reinforced from the left column, there is a tendency to over-estimate the new figure added in the quotient.

Example.—Find the cube root of 31875·8972.

		31875·8972 (31·3727
		27
90	2700	4875
1	91	
91	2791	2791
930	288300	1084897
3	2799	
933	291099	873297
		211600
		203769
		7831
		5822
		2009

Ans. 31·373

VI. The student who has done trigonometry is advised to use logarithm tables freely, but any explanation would be beyond the scope of this work. In many results, corrections for differences will be found unnecessary, and five-figure logarithms will be generally sufficient.

VII. More errors occur from avoidable blunders in calculation than from any other cause. Care should be taken to note whether the radius or diameter of a circle is given; to distinguish between height and slant height in a cone; to avoid confusion between yards, feet, and inches; to be clear whether a magnitude is measured in linear, square, or cubic measure.

All areas and surfaces must be expressed in square, and all volumes in cubic measure. The former involve the square of the unit of length, and the latter the cube. Observation of the dimensions of the units in various expressions form a useful check on the accuracy of reasoning and result.

VIII. It is well to decide at the outset on the unit of measurement which is most convenient under the circumstances, and to employ it rigorously throughout.

IX. Short division and the use of decimals generally conduce to a saving of labour. Such answers as £$581\frac{1972}{8953}$ and $518\frac{4321}{6927}$ gals. should be avoided. The former should be given as £581 4*s.* 4·8*d.*, and the latter as 518·623 gals.

X. Extreme accuracy in an example when an approximate value of π, such as $\frac{22}{7}$, has been taken, is an absurdity. This value of π introduces an error of 1 in 2500. In all cases the student should use his judgment as to whether he is exceeding the degree of accuracy required. The use of recurring decimals in an approximate answer should be avoided.

XI. If an answer is palpably absurd in its character, it is best to glance quickly through the previous work to find the mistake. If the error is not obvious, or if it is in an early stage of the calculation, it is nearly always quicker to recommence the question entirely afresh, and, if possible, to vary the method employed.

HARD EXERCISES.

1. A sheet of paper measures 18 in. by 8 in.; it is rolled into cylinders with the length and breadth alternately forming the perimeter: compare the volumes of the two cylinders thus formed.

2. What sized pipe should be used with a 6-in. pipe to take off the flow from a 10-in. pipe?

3. If a mile equals 1609·33 metres, find the number of square metres in $7\frac{1}{2}$ ac.

4. The number of cubic feet in a cubical box is half the number of square feet of surface: find the volume of the box.

5. Find the amount of copper required for a submarine cable, if the length be 500 miles, and the thickness of each of the five strands of the central wire be $\frac{1}{8}$ in.

6. Water flows from an inch pipe with the velocity of 4 ft. a second: find the discharge in gallons per hour. (1 cub. ft. of water weighs 1000 ozs.; 1 gal. of water weighs 10 lbs.)

7. A spherical lump of metal 12 in. in diameter is cast into a conical mould whose height is 9 in.: find the diameter of the base of the cone.

8. Divide a straight line 12 ft. long into two parts, so that the square on one part equals the rectangle contained by the whole line and the other part.

9. A stack is to be built on a rectangular base measuring 25 ft. by 18 ft., and it is to be 10 ft. to the eaves and 18 ft. to the top: find the difference between the contents when the top ends in an edge from end to end of the stack, and when it goes to a point.

10. A cylindrical vessel full of water is 10 in. high and 4 in. in diameter: how high will it fill a cube with an 8-in. edge?

11. Three posts of equal height are placed in a row at intervals of a mile along a straight canal: prove that the middle post is 8 in. higher than the line joining the tops of the other posts. (Radius = 4000 miles.)

12. Find approximately the length of paper which can be rolled tightly into a cylinder 6 in. in diameter, if the paper is 0·004 in. thick.

13. A triangular area on a map has its sides 2·5, 1·7, 1·6 in.; the scale is 6 in. to a mile: find the area of the triangular tract in acres.

14. A thin metal sheet, 2 ft. square, has equal squares cut from the corners, and the flaps are bent up to form an open box; the four squares are joined up and just form a lid: find the volume of the box.

15. A balloon is hemispherical above and conical below; its diameter is 18 ft. and its total height is 21 ft.: find its capacity in cubic feet.

16. Which are the cheaper—cylindrical candles 1 in. in diameter and 6 in. long at 40 a shilling, or candles of the same material $\frac{3}{4}$ in. in diameter and 8 in. long at 3 a penny?

17. A segment of a sphere is 2 in. high, and its base is 10 in. in diameter: find its volume and total surface.

18. Find the volume of a gas-pipe 10 ft. long, with external and internal radii of 1 in. and $\frac{1}{2}$ in.

19. Two circles of radii 8 in. are described as in Euc. I. 1: find the area included between them.

20. Two points on a line of metals on a railway are 20 yds. apart, and a point on the rails midway between them is 10 in. from the line joining them: find the radius of the curve.

21. Find the error per cent. in taking $\pi = \frac{22}{7}$ instead of 3·14159..., (1) in finding the area of a circle, (2) in finding the volume of an anchor ring.

22. A cone, whose slant height is 1 ft. and diameter 8 in., has its vertex fixed, and is then rolled round on a table: how many times will the cone turn on its axis when it has made 100 revolutions round its vertex?

23. Find the height of a right cone in a spherical shell of 5 in. radius,

if the diameter of the base of the cone is 6 in., and the vertex touches the top. Deduce the volume between the sphere and cone.

24. Find the volume of a regular tetrahedron * if each edge measures 2 in.

25. A cylindrical vase is 8 in. high and 4 in. in diameter, but a conical projection rises from the base, and this is 2 in. high and 4 in. in diameter: find the cubical contents of the vase.

26. A pony is tethered by a rope to a corner of a triangular field, and the adjacent sides are 160 yds. and 200 yds.; the sides include half a right angle, and the rope is 40 yds. long: find the area of the field ungrazed by the pony.

27. The weight of a cylindrical hollow lead pipe of external diameter 1 in. and of length 1 ft. was found to be just half that of a solid lead cylinder of external diameter $\frac{3}{4}$ in. and length 6 in.: find the internal radius of the pipe.

28. Find approximately the number of tons of chalk which would be removed in making the Channel Tunnel, if the length were 23 miles and the section semicircular and 6 ft. in radius. (Density of chalk = 2·4; 1 cub. ft. of water weighs 62·5 lbs.)

29. Find the length of the groove in a rifle, if it makes one complete revolution in a barrel $2\frac{1}{2}$ ft. long and $\frac{1}{2}$ in. in diameter.

30. A square court is 150 ft. square, and paths 6 ft. wide run round the court and across the diagonals: find the cost of paving these paths at 4*s.* a square yard.

31. A river, 14 ft. deep, 182 yds. wide, flows at the rate of 3 miles an hour: how many gallons of water pass any given point in a minute? How many tons go to the sea in a year? (1 cub. ft. = $6\frac{1}{4}$ gals.; 1 cub. ft. weighs 1000 ozs.)

32. A 10-in. circular pipe empties a reservoir in four hours: if the reservoir is 10 ft. deep and measures 50 ft. by 40 ft., find the velocity of the water in feet per second.

33. A sector of a circle subtends 60° at the centre of a circle of 12 in. radius: find the volume of the cone into which the paper can be rolled when the extreme radii are joined.

34. A cylinder is 4 in. in diameter and stands on a circular base; it is cut obliquely at the top, so that the greatest and least heights of the section are 8 in. and 11 in.: find the volume and total surface.

35. Find approximately the distance a man could see from the top of a mountain 4000 ft. high, if the radius of the earth is assumed to be 4000 miles.

36. A sheet of paper in the shape of an equilateral triangle has its

* A regular tetrahedron is a pyramid on a triangular base, and each face is an equilateral triangle.

corners folded up so as to form a regular tetrahedron: if each side of the paper was 12 in. long, find the volume of the tetrahedron.

37. Six equal circles of 1 in. diameter have their centres on a given circle, and each of the six touches the two neighbouring circles: find the area contained by their inner circumferences.

38. A sheet of metal in the shape of a regular hexagon, whose sides are 8 in., has triangular notches cut at each corner, and the flaps are bent up to form an open hexagonal box, whose sides are 1 in. high: find the amount of metal wasted, and the volume of the box.

39. Find the amount of silk required to make a parachute in the form of a segment of a sphere, if the radius of the base is 12 ft. and the height 6 ft.

40. Find the radius of a sphere whose volume is 1 cub. ft.

41. Find the velocity at which a person is travelling in latitude 45° owing to the earth's rotation. (The radius of the earth may be assumed to be 4000 miles.)

42. Five hundred spherical bullets, $\frac{1}{2}$ in. in diameter, are cast into a cylinder a foot high: find the radius of the cylinder; find also the dimensions of the cone into which the bullets could be recast if the height equals the diameter of the base.

43. Find the edge of the greatest cube which can be placed with its sides vertical under a cone 12 in. high and 16 in. in diameter.

44. A frustum of a cone is 6 in. high, and has the diameter of the ends 4 in. and 12 in.; it is placed on a table, and rolls without slipping until it returns to its former position: find the area over which it has rolled.

45. Find the volume of the greatest sphere which can be placed under a pyramid on a square base with every edge 1 ft. long.

46. Three hemispherical soap-bubbles join into one: find its diameter, if the original bubbles were 2 in., 3 in., and 4 in. in diameter.

47. Find the cosine of the angle, and the angle between two faces of a regular tetrahedron.

48. A rectangular pathway measures 787·4 metres by 1·526 metre, and costs $1\frac{1}{2}$ franc per sq. metre: find the length of another pathway in yards which costs the same as the above, and whose breadth is 6 ft., and which is made at the rate of 1*s* $1\frac{1}{2}$*d.* per sq. yd. (1 metre = 39·37 in.; £1 = 25 francs.)

49. The corners of a block of soap in the shape of a cube whose edges are 3 in. are cut off so that the edges are cut 1 in. from every corner: find the volume of the remainder.

50. A circular earthwork has a semicircular cross-section, and the internal and external circumferences are 130 ft. and 180 ft.: find the number of cubic feet of earth. What would be the depth of the trench from which the earth was thrown up, if it were 5 ft. wide and of rectangular cross-section, and just outside the mound?

51. Find the whole surface and volume of a solid figure formed by cutting from a sphere two segments by parallel planes 4 in. from the centre. (The diameter of the sphere is 10 in.)

52. Prove that the following figure is a frustum of a wedge and not of a pyramid, and find its volume without assuming the formula for such a frustum: rectangular base, 40 ft. by 30 ft.; uniform height, 12 ft.; rectangular top, 30 ft. by 20 ft.

53. The difference between the area of a square and its inscribed circle is 21·35 sq. in.: find the side of the square.

54. How many square miles of ocean are visible to a man 20 ft. above its surface, if the radius of the earth be 4000 miles?

55. At what distance will a line of white cliff 40 ft. high be just visible to a man 10 ft. above the sea?

56. A cylindrical block of wood 1 ft. in diameter and 10 in. high has its ends hollowed in the shapes of segments of spheres to a depth of 2 in. at each end: find the volume of the block.

57. A convex lens is 2 in. thick and each surface has a radius of 13 in.; the diameter of the lens is 10 in.: find its volume and surface.

58. As large ellipses as possible are stamped from rectangular sheets of tin measuring 2 ft. by 3 ft.: find the percentage of waste, if the principal axes of the ellipse are parallel to the sides of the rectangle.

59. Find the number of square miles between the 30th and 45th parallels of latitude, assuming the earth's radius to be 4000 miles.

60. Find the radius of a hemisphere whose volume is 9 cub. in.

61. A regular tetrahedron has an edge of 4 in. length: find the volumes of the inscribed and circumscribed spheres.

62. The distance of the sun is about 95,000,000 miles, and the earth moves round the sun in an ellipse: if the earth moved round in a circle, with the sun at the centre, what would be our velocity in miles a second?

63. Three conical extinguishers are placed on a table with their bases touching, and a larger cone is placed over them: if the latter cone just fits, find the height and base of the cone. (The small cones are 3 in. high and 1 in. in diameter.)

64. A hollow cone is 12 in. high, and its base is 16 in. in diameter; a block on a square base, and twice as tall as it is wide, is just able to stand inside the cone: find the base of the block.

65. A ship running west in latitude 60° finds that the time has changed 50 minutes: find the number of miles the ship has travelled.

66. A cylindrical vat is to be made having its height twice its diameter, and it is to contain as much as twenty 36-gal. casks: find the radius, given that 1 gal. of water weighs 10 lbs., and that 1 cub. ft. of water weighs 1000 ozs.

SANDHURST QUESTIONS.

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1. *Dec.* 1883.—Six iron tubes, the cross-section of any one of them being a square, are joined together so as to form a hexagonal ring, which can be filled with water: if the area of the cross-section be 16 sq. in., and the distance between any two internal opposite corners of the hexagon be 1 yd., find the whole surface of the tubing and its contents, neglecting the thickness of the sheet iron.

2. *Dec.* 1883.—A cylindrical tower 24 ft. in diameter and 30 ft. high is capped with a hemispherical dome; the top of the dome is cut off, and over the orifice formed is built a cylindrical lantern 8 ft. in diameter and 10 ft. high, closed at the top by a plane surface: find the whole exterior surface of the building.

3. *Dec.* 1890.—In the pentagonal field ABCDE, the length of AC is 50 yds., and the perpendiculars from B, D, and E upon AC are 10, 20, and 15 yds., the distances from A to the feet of the perpendicular from D and E being 40 and 10 yds.: find the area.

4. *Dec.* 1890.—A sphere of 1 ft. radius rests on a table: find the volume of the right hollow cone which can just cover it, the section of the cone through the axis being an equilateral triangle.

5. *Dec.* 1892.—The minute hand of a clock is 10 in. long: find the area on the clock face which it describes between 9 a.m. and 9.35 a.m.

6. *Dec.* 1892.—From a cubic foot of lead is cut out a pyramid whose base is one face of the cube, and whose vertex lies in the face opposite the base: if the remainder of the lead is melted and cast into a sphere, find its radius.

7. *Dec.* 1888.—The area of an equilateral triangle is 17320·5 sq. ft.; about each angular point, as centre, a circle is described with radius equal to half the length of a side of the triangle: find the area of the space included between the three circles. ($\pi = 3{\cdot}1416$; $\sqrt{3} = 1{\cdot}73205$.)

8. *Dec.* 1888.—A hollow cone, the length of whose slant height is twice the radius of the base, is held with its vertex vertically downwards, and completely filled with water; a sphere of greater density than water is gradually immersed, and it is found that, when it rests upon the sides of the interior of the cone, it is just submerged: find the amount of water displaced by the sphere, and also the amount contained between the sphere and the vertex of the cone. (Consider radius of the base of cone as 1·73205 in.)

9. *Dec.* 1889.—Find the expense of paving a circular court 80 ft. in diameter, at 3*s.* 4*d.* per sq. ft., leaving in the centre a space for a fountain, in the shape of a hexagon, each side of which is 1 yd.

10. *June,* 1890.—Prove that the area of a trapezoid is one-half the product of the sum of the two parallel sides by the perpendicular distance

between them. The area of a trapezoidal field is $4\frac{1}{2}$ ac.; the perpendicular distance between the parallel sides is 120 yds.; and one of the parallel sides is 10 ch.: find the other.

11. *June*, 1890.—Express the volume of a cone in terms of the radius of the base and the vertical height. If the diameters of the circular ends of a frustum of a cone be 4 in. and 6 in., and the volume of the frustum be 209 cub. in., find the height of the cone. ($\pi = 3\frac{1}{7}$.)

12. *June*, 1891.—105 halfpenny pieces lying on a flat surface with their edges in contact are just contained by a frame in the form of an equilateral triangle: the diameter of a halfpenny being 1 in., show that the side of the triangle is $(13 + \sqrt{3})$ in., and calculate its area approximately.

13. *June*, 1891.—Gold is 19·25 times as heavy as water, and 1 cub. ft. of water weighs 997 ozs. avd.: find (approximately) how many square feet 1 cub. in. of gold will cover in the form of gold leaf, given that 1 gr. of gold will cover 56 sq. in.

14. *Dec.* 1891.—Find the area of a triangle, whose sides are 13·6, 15, and 15·4 in. Also find (correct to the thousandth part of an inch) the length of one of the equal sides of an isosceles triangle, on a base of 14 in., having the same area.

15. *Dec.* 1891.—A circular room, surrounded by a hemispherical vaulted roof, contains 5236 cub. ft. of air, and the internal diameter of the building is equal to the height of the crown of the vault above the floor: find the height, assuming 3·1416 to be the value of π.

16. *July*, 1889.—Assuming that $\pi = 3\cdot1416$, find the perimeter and the radius of a circle, the area of which is 5·309304 sq. ft.

17. *July*, 1889.—A solid sphere fits closely into the inside of a closed cylindrical box, the height of which is equal to the diameter of the cylinder: having given the radius of the sphere, write down the expressions for the volume of the sphere, the surface of the sphere, and the volume of the empty space between the sphere and the cylinder. If the volume of this empty space is 134·0416 cub. in., what is the radius of the sphere?

18. *July*, 1892.—Two pipes, one of lead and the other of tin, are respectively 49 and 61·6 in. long; they both have the same internal diameter, 1 in.; and the external diameter of the lead pipe is 1·2 in.: if lead is eleven times and tin seven times as heavy as water, what must be the external diameter of the tin pipe, that both pipes may have the same weight?

19. *July*, 1892.—Assuming a drop of water to be spherical, and $\frac{1}{10}$ in. in diameter, to what depth will 500 drops fill a conical wine-glass, the cone of which has a height equal to the diameter of its rim?

20. *Dec.* 1889.—A right prism on a triangular base, each of whose sides is 21 in., is such that a sphere, described within it, touches its five faces: find the volume of the sphere, and of the space between it and the surface of the prism. ($\pi = \frac{22}{7}$; and $\sqrt{3} = 1\cdot732$.)

WOOLWICH QUESTIONS.

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1. *July*, 1883.—A flagstaff 100 ft. high stands in the centre of an equilateral triangle, which is horizontal; from the top of the flagstaff each side of the triangle subtends an angle of 60°: find a side of the triangle.

2. *July*, 1888.—The extremity of the shadow of a flagstaff 6 ft. high, standing on the top of a regular pyramid on a square base, just reaches a side of the base, and is distant 56 ft. and 8 ft. from the extremities of that side: if the height of the pyramid be 34 ft., find the sun's altitude.

3. *Nov.* 1888.—Determine the diameter of a cylindrical gasholder to contain 10,000,000 cub. ft. of gas, supposing the height to be made equal to the diameter; and determine in tons the weight of iron plate, weighing $2\frac{1}{2}$ lbs. per square foot, required in the construction of the gasholder, supposing it open at the bottom and closed by a flat top.

4. *Nov.* 1888.—Determine the number of cubic yards in a bank of earth on a horizontal retangular base 60 ft. long and 20 ft. broad, the four sides of the bank sloping up to a ridge at an angle of 40° to the horizon.

5. *Nov.* 1884.—A solid cube of lead weighs 126·44 lbs.; 998 ozs. of water occupy 1 cub. ft., and 1 cub. ft. of lead is 11·352 times as heavy as 1 cub. ft. of water: find by logarithms the length of a side of the cube of lead to six places of decimals of a foot.

6. *June*, 1891.—A railway tunnel consists of a hollow semi-cylindrical top, terminated below in a trough with slanting sides and flat base: the radius of the former being 12 ft., the base and height of the latter being 20 ft. and 18 ft. respectively, and the length of the tunnel 1200 yds., find the cost of facing the sides and roof with brick at 1*s.* 6*d.* per square foot.

7. *June*, 1891.—Within a hollow sphere of 1 ft. radius is placed a right prism, the ends of which are equilateral triangles: the side of one of these being 1 ft. in length, and the surface of the sphere being in contact with all the six angular points of the prism, find, in cubic inches, the volume of the latter.

8. *Nov.* 1892.—Find the area of the surface (including the ends) of a hexagonal prism, whose height is 8 ft., the base being a regular hexagon with a side of length 3 ft.

9. *Nov.* 1892.—The radii of the internal and external surfaces of a hollow spherical shell of metal are 3 ft. and 5 ft. respectively: if it be melted down, and the material formed into a cube, find an approximate value for the length of an edge of the cube.

10. *Dec.* 1891.—A piece of wood is in the form of a regular pyramid on a square base; the side of the base is 6 in., and the perpendicular distance of the vertex from the base is 8 in.: find the number of cubic

inches in the volume of the wood, and the number of square inches in its surface.

11. *Dec.* 1891.—A cylindrical boiler is hemispherical at its two ends; its radius is 2 ft., and its total length is 8 ft.: assuming that 1 cub. ft. of water weighs 62·5 lbs., find the number of tons of water which will fill the boiler. (Take $\pi = 3·14$.)

12. *July,* 1892.—The height of a conical tent is $7\frac{1}{2}$ ft., and it is to enclose 200 sq. yds. of ground: find how much canvas will be required. ($\pi = \frac{22}{7}$.)

13. *July,* 1892.—The silk covering of an umbrella forms a portion of a sphere of $3\frac{1}{2}$ ft. radius, the area of the silk being $14\frac{2}{3}$ sq. ft.: find the area of the ground sheltered from vertical rain when the stick is held upright. ($\pi = \frac{22}{7}$.)

14. *June,* 1890.—How many square yards of canvas are required to make a conical tent 9 ft. high, such that a man of 6 ft. could stand anywhere inside, within a radius of 2 ft. from the centre without stooping?

15. *June,* 1890.—A pint standard is in the form of a frustum of a circular cone; its height is $4\frac{1}{2}$ in., and the diameter of its base is $3\frac{1}{2}$ in., both measurements taken inside: find the diameter of the top, being given that a gallon of water weighs 10 lbs., and 1 cub. ft. of water weighs 1000 ozs.

ANSWERS.

EXERCISE I.

1. 82195 yds.
2. 476 miles 5 fur. 37 po. $3\frac{1}{2}$ yds.
3. 77880 ft.
4. 350 ch.
5. 9 miles 4 fur. 20 po.
6. 341 yds.
7. 1070 ch.
8. 110 miles 3 fur. 33 po. 2 yds. 2 ft. 10 in.
9. 4730 yds.
10. 330 ft.
11. $3\frac{3}{4}$ miles.
12. 415 ch.

EXERCISE II.

1. $86938\frac{1}{2}$ sq. yds.
2. 187 ac. 0 ro. 9 po. $18\frac{3}{4}$ sq. yds.
3. 163350 sq. yds.
4. 7 ac. 2 ro. 29·75 po.
5. 675000 sq. lks.
6. 54 ac. 3 ro. 24 po.
7. 1575000 sq. lks.
8. 53 ac. 3 ro. $25\frac{1}{4}$ po.
9. 123420 sq. yds.
10. 44 sq. yds. 7 sq. ft. 97 sq. in.
11. $144443\frac{3}{4}$ sq. yds.
12. 40 ac. 2 ro. 0·8 po.
13. 70 ac. 3 ro. 32 po.
14. 5 ac.
15. 10.
16. 200 sq. yds.

EXERCISE III. (A).

1. 27 sq. ft. 17 sq. in.
2. 26 sq. ft. 142 sq. in.
3. 16 sq. ft. 81 sq. in.
4. 273 sq. ft. 63 sq. in.
5. 396 sq. ft. 60 sq. in.
6. 28 sq. ft. 60 sq. in.
7. 238 sq. ft. 90 sq. in.
8. 31 sq. ft. 87 sq. in.
9. 21 sq. yds. 5 sq. ft. 36 sq. in.
10. 70 sq. yds. 7 sq. ft. 76 sq. in.
11. 144 sq. yds. 3 sq. ft. 90 sq. in.
12. 35 sq. yds. 5 sq. ft. 48 sq. in.

EXERCISE III. (B).

1. 3 ft. 10 in.
2. 4 ft. 6 in.
3. 15 ft. 6 in.
4. $269\frac{1}{2}$ yds.
5. 70 yds.
6. 352 yds.
7. $36\frac{3}{32}$ yds.
8. 6 ch. 25 lks.
9. 18 yds. 2 ft.
10. 10 yds. 1 ft.
11. $23\frac{3}{5}$ ft.
12. 20 in.

EXERCISE IV.

1. 4 ft. 3·22 in.
2. 8545 ft.
3. 2 ch. 73·8 lks.
4. 22·456 yds.
5. 5 ft. 8 in.
6. 5·16 fur.

7. 333 ft. **8.** 2 ch. 96·5 lks. **9.** 29·16 in.
10. 600·2 miles. **11.** 205 yds. **12.** 20 ft.
13. 90 ft. **14.** 50 ft. **15.** 9·949 ft.
16. $20\frac{2}{3}$ ft.

EXERCISE V.

1. $42\frac{1}{4}$ sq. yds. **2.** 169 sq. yds. **3.** $915\frac{1}{16}$ sq. yds.
4. $765\frac{4}{9}$ sq. yds. **5.** $113\frac{7}{9}$ sq. yds. **6.** $27\frac{22}{81}$ sq. yds.
7. 160 sq. yds. 4 sq. ft. **8.** 205 sq. yds. 4 sq. ft.
9. 11 sq. yds. 1 sq. ft. **10.** 5 ac. 2 ro. 38 po. $7\frac{1}{2}$ yds. 7 ft.
11. 2 ac. 2 ro. **12.** 3 ac. 0 ro. 4 po. **13.** 15 ac. 0 ro. 1 po.
14. 33 ac. 2 ro. 33·43 po. **15.** 105 ac. 2 ro. 20 po.
16. 69 ac. 1 ro. 29·16 po. **17.** 93 yds. **18.** 173·9 yds.
19. $229\frac{1}{2}$ yds nearly. **20.** 8·004 ft. **21.** 37812·5 sq. yds.
22. 1548800 sq. yds. **23.** 5 sq. yds. 2 sq. ft. 76 sq. in.
24. 5 ac. **25.** 3·742 in. **26.** 55·1 yds.
27. 220 yds. **28.** 120·5 yds. nearly. **29.** $28\frac{1}{2}$ yds.
30. 99 yds. **31.** £5 6*s.* 2*d.* **32.** 7 ac. 2 ro. 13·8 po.
33. 10·488 ft. **34.** 23 ac. 2 ro. 34·56 po.
35. 220 yds. **36.** 57 yds. **37.** £2 3*s.* 9*d.*
38. £27 10*s.* **39.** 1071·6 lks. **40.** 479·2 yds.
41. 3 ft. and 14 ft.

EXERCISE VI.

1. 18 sq. yds. 8 sq. ft. **2.** 127 sq. yds. 4 sq. ft.
3. 40 sq. yds. 8 sq. ft. **4.** 4 sq. yds.
5. 36 sq. yds. 7 sq. ft. 72 sq. in. **6.** 61 sq. yds. 1 sq. ft.
7. 5 ac. 2 ro. 34·6368 po. **8.** 13 ac. 1 ro. 22·72 po.
9. 9 ac. 1 ro. 14 po. **10.** 44 yds. **11.** 1100 yds.
12. 1 ch. 64 lks. **13.** 46 ft. 10 in. **14.** 53 ft. 6 in.
15. 56 ft. **16.** 15 sq. yds. 1 sq. ft. 24 sq. in.
17. 19 sq. yds. 7 sq. ft. 18 sq. in. **18.** 21 sq. yds. 6 sq. ft. 95 sq. in.
19. 49 sq. yds. 3 sq. ft. 132 sq. in. **20.** 62 sq. yds. 3 sq. ft. 108 sq. in.
21. 71 sq. yds. 5 sq. ft. **22.** 200 yds. **23.** 567 lks.
24. Length, 24 ft.; breadth, 18 ft.; height, 12 ft. **25.** 1*s.*
26. £95; £22 15*s.* **27.** £5 5*s.* **28.** 17·8885 ft.
29. $133\frac{7}{18}$ yds. **30.** £16 13*s.* 10*d.* **31.** 192 slabs. **32.** 20 ft.
33. (i.) $63\frac{7}{12}$ sq. yds.; (ii.) $20\frac{35}{72}$; (iii.) £6 2*s.* 11*d.*
34. 47·319 yds., and £10 12*s.* 11*d.* **35.** 852 sq. ft. 50 sq. in.
36. (i.) $21\frac{7}{72}$ sq. yds.; (ii.) $67\frac{5}{6}$ sq. yds.; (iii.) $28\frac{7}{54}$ yds.; (iv.) £2 19*s.* 3*d.*; (v.) £3 17*s.* $4\frac{5}{18}$*d.*
37. (i.) $69\frac{22}{54}$ sq. yds.; (ii.) $20\frac{10}{27}$ sq. yds.; (iii.) $17\frac{29}{63}$ yds.; (iv.) £1 6*s.* $0\frac{5}{12}$*d.*; (v.) £2 16*s.* $0\frac{2}{9}$*d.*; (vi.) £1 10*s.* $6\frac{2}{3}$*d.*

EXERCISE VII.

1. 1 ac. **2.** 364 sq. yds. **3.** 10 ft.
4. 1 sq. ft. $43\frac{1}{2}$ sq. in. **5.** 70 yds. **6.** $3\frac{1}{2}$ ac.

7. 15 yds. 2 ft.
8. 498·8 sq. ft.
9. 124 sq. ft. 8 sq. in.
10. 31 ac. 0 ro. 25·3 po.
11. £3 14*s.* 1*d.*
12. $6\frac{3}{7}$.
13. 1 sq. ft. 116·96 sq. in.
14. 4056 sq. ft.
15. 33 ft.
16. 40 ft.
17. $149\frac{2}{9}$ yds.
18. 173 yds. 1 ft.

EXERCISE VIII.

1. 212·5 sq. ft.
2. 8 ac. 2 ro. 25·568 po.
3. 59 sq. ft. 4 sq. in.
4. 40 ac. 3 ro. 21·344 po.
5. 25 yds.
6. 31 ch. 78 lks.
7. 89·3.
8. 991·483.
9. 515212·33.
10. $\frac{3}{10}\sqrt{2}$.
11. £5 11*s.* $6\frac{3}{4}$*d.*
12. 1428.
13. 40·08.
14. 35·07 sq. yds.
15. 400 lks.
16. £8 3*s.*
17. £3 5*s.* $6\frac{3}{5}$*d.*
18. 561·168 sq. ft.
19. £1 4*s.* $2\frac{2}{5}$*d.*
20. 46·153 ft.
21. 22 ch.
22. £7 18*s.* 7*d.*
23. £843 7*s.* 8*d.*
24. 6·6143 ch.
25. 1 ac. 3 ro. 32 po. 2 sq. yds. 6 sq. ft.
26. As $\sqrt{3} : 2$.

EXERCISE IX.

1. $418\frac{1}{9}$ sq. yds.
2. $18\frac{1}{2}$ in.
3. 240 sq. in.
4. 1800 sq. yds.
5. 3 ro. 30 po. 18 yds. $\frac{1}{2}$ ft.
6. 1 ro. 11 po. 25 yds. 3·7 sq. ft.
7. 23 po. $5\frac{3}{4}$ sq. ft.
8. £121.
9. 2 ac. 3 ro. $28\frac{12}{121}$ po.
10. £2 7*s.* 3*d.*
11. 7 ac. 2 ro. $13\frac{1}{2}$ po.
12. £24 6*s.* $7\frac{1}{4}$*d.*
13. 27 ac. 1 ro. 24·16 po.
14. 126 ac.
15. 3 ac. 11 po. $17\frac{1}{4}$ yds. 7 sq. ft.
16. 125 ft.
17. 625 ft.
18. 6 ac. 3 ro. 37 po. 10 yds. 6 ft. 108 sq. in.
19. 2 ac. 2 ro. 36 po.
20. 98 sq. yds. $3\frac{1}{3}$ sq. ft.

EXERCISE X.

1. 210·438 sq. in.
2. 1635 sq. yds. 7·9 sq. ft.
3. 275 sq. ft. 90 sq. in.
4. 2338·272 sq. ft.
5. £97 19*s.* $4\frac{1}{2}$*d.*
6. $2\sqrt{2}$ sq. ft.
7. 12·4248 ft.
8. 10·392 ft.
9. As 225 : 233.
10. 6495 sq. ft.
11. 5·177 ft.
12. 129·9 sq. ft.
13. 8·66 ft.

EXERCISE XI.

1. 1 ac. 1 ro. 32 po.
2. 1 ac. 1 ro. 14·4 po.
3. 3 ac. 1 ro. 4 po.
4. 3 ro.
5. 3 ac. $29\frac{1}{2}$ po.
6. 15 ac. 0 ro. 26 po.
7. 1 ro. 9·7 po.
8. 2 ro. 26·9 po.
9. 3 ac. 1 ro. 14·6 po.
10. 4 ac. 3 ro. $24\frac{1}{2}$ po.
11. 1 ac. 1 ro. 32 po.
12. 1 ac. 1 ro. 18 po.
13. 1 ac. 18·64 po.
14. 3 ac. 2 ro. 12 po.
15. 2 ac. 1 ro. 22 po.
16. 13 ft. and 546 sq. ft.
17. 126 ac.
18. 17 ac. $3\frac{1}{2}$ ro.

EXERCISE XII.

1. 84, 80, 74.
2. $DE = AC \times \frac{2\sqrt{2}}{\sqrt{17}}$.
3. As 9 : 4.
4. 98·663 ft.
5. 68·3 sq. in.
6. $9\sqrt{6}$, $9\sqrt{3}$.
7. As 1 : 36.
8. As 1 : 660.
9. 11·832, 17·748.
10. 115 ft.
11. $\frac{10}{3}\sqrt{2}$.
12. As 1 : 11200.
13. $2006\frac{3}{4}$ sq. ft.
14. As 3 : $2\sqrt{3}$.
15. 92·16 sq. in.

XIV.—MISCELLANEOUS EXERCISES.

I.

1. 1 ac. 1 ro. 6 po. $10\frac{3}{4}$ sq. yds.
2. $6\frac{3}{4}$ ac.
3. 149 yds. 8 in.; £33 11*s*. 6*d*.
4. 800.
5. 3108 yds.
6. 2 ac. 2 ro. 30·06 po.

II.

1. $396\frac{5}{12}$ ft.
2. 7 ft.
3. 78 yds.; 58 yds.
4. £5 10*s*.
5. 1460.
6. $14\frac{1}{16}$ ft.

III.

1. 31·5; 952·56.
2. $60\frac{2}{3}$ yds.
3. £36 11*s*. 6*d*.
4. £24 10*s*.
5. 15 ac.
6. 4 ac. 3 ro. 16·24 po.

IV.

1. $82\frac{2}{3}$ yds.
2. 10·39 ins.
3. £196 10*s*.
4. 77 yds. 2 ft. 11 in.
5. 4861·02.
6. £23 4*s*.

V.

1. 155·56 ft.
2. £10 10*s*.
3. 54·45 sq. in.
4. $12\frac{1}{4}$ ch.; $8\frac{3}{4}$ ch.
5. £1 5*s*. $3\frac{3}{4}$*d*.
6. £5 11*s*. $6\frac{3}{4}$*d*.

VI.

1. 98 yds.
2. 5 ft. $2\frac{1}{2}$ in.
3. 10 ac. 3 ro. 9 po. 28·05 yds.
4. $66\frac{2}{3}$ ft.; 136 ft.; $197\frac{1}{3}$ ft.
5. £176 17*s*. 6·28*d*.
6. 102·46 miles.

VII.

1. £8 6*s*. 6*d*.
2. 498·8 sq. ft.
3. As 1 : 1214.
4. 20 ac. 3 ro. 32 po. 2 yds.
5. 19*s*. 2*d*.
6. $147\frac{7}{8}$ sq. ft.

VIII.

1. $\frac{1}{4}$ in. to the mile.
2. £12 19*s*. $9\frac{3}{16}$*d*.
3. 46·188 ft.
4. As 2 : $\sqrt[4]{27}$.
5. 242 yds.
6. 3 ac. $30\frac{1}{2}$ po.

IX.

1. 595 sq. ft. 2. £2 15s. 3. 1054 ft.; 625 ft.
4. 1039·23 sq. ft. 5. 40 ft. 6. 3 ac. 0 ro. 12 po.

X.

1. 2144 sq. ft. 2. 9 ft. 3. 1152 sq. yds.; £19 4s.
4. 236·4 yds. 5. $5(\sqrt{2}+1)$. 6. 1 ac. 18·64 po.

EXERCISE XIII.

1. $3\sqrt{2}$ ft. 2. 5·38 ft. 3. 2·6 ft. nearly 4. 1·155 ft.
5. 3·456 ft. 6. $3\sqrt{2}$ ft. 7. 72 sq. yds. 8. 8·5 ft.
9. 20 sq. ft.

EXERCISE XIV.

1. 8·23 in. 2. 50·7 ft. 3. 9·01 in. 4. 31·81 ft.
5. 5 in. 6. 9 ft. 7. 2 ft., 18 ft. 8. 29 ft.
9. 22·628 in. 10. $5\sqrt{7}$ ft. 11. 108 in. 12. 10·12 in.
13. 80. 14. $150\frac{1}{4}$ ft. 15. $29\frac{83}{113}$.

EXERCISE XV.

1. $47\frac{1}{7}$ ft. 2. 5 ft. 6 in. 3. 17 yds. $10\frac{2}{7}$ in.
4. 14 ch. 92 lks 5. 125·644 in. 6. 21 ft. 5·6 in.
7. 2827·44 yds. 8. 84·8232 ft 9. 35 yds. 10. 280 yds.
11. $15\frac{10}{11}$ ft. 12. ·222 in. 13. ·3183 ft. 14. 70·028 yds.
15. 114·59 ft. 16. 1·27 in. 17. $4\frac{2}{3}$ ft. 18. 480.
19. 2 ft. 4·9 in. 20. As 4 : π. 21. As π : 3. 22. 31′ 50″.
23. $4\frac{7}{22}$ min. 24. $1571\frac{3}{7}$ ft. 25. 210. 26. As 86 : 85 nearly.

EXERCISE XVI.

1. 35·81°. 2. 114° 35′ 30″ 3. 4° 46′ $10\frac{10}{11}$″.
4. (i.) 22° 55′ 6″ (ii.) 57° 17′ 45″; (iii.) 38° 11′ 50″; (iv.) 128° 54′ 56·25″. 5. (i.) $\frac{\pi}{4}$; (ii.) $\frac{5\pi}{6}$; (iii.) $\frac{5\pi}{8}$.
6. $38\frac{2}{3}$ in. 7. 26 ft. 8. 32 ft. 9. 17·75 in.
10. 20°. 11. $8\frac{2}{11}$ after 3, and $24\frac{6}{11}$ after 3. 12. 28° 38′ 52″.
13. As 3 : 2. 14. 6·117 in. 15 $21\frac{1}{3}$ ft. 16. 76° $21\frac{9}{11}$′.
17. $10\frac{1}{2}$ in. 18. $4\frac{1}{30}$ in. 19. 126° 49′. 20. 1·8849.
21. 7 ft. 22. 28° 38′ 52·5″. 23. $167\frac{2}{3}$°. 24. $4\frac{1}{6}$ in.

EXERCISE XVII.

1. 5544 sq. ft. 2. 125 ac. 2 ro. 34 po. $8\frac{9}{14}$ sq. yds.
3. $31428\frac{4}{7}$ sq. ft. 4. $20114\frac{2}{7}$ sq. ft. 5. 644·598 sq. in.
6. 9 ac. 2 ro. 39·76 po. 7. 29 ac. 20 po. 8. 4476375 sq. yds.
9. 1963·5 sq. ft. 10. 70 ac. 3 ro. 35 po. $14\frac{1}{4}$ sq. yds. 8·46 sq. ft.

11. 125 ac. 2 ro. 26 po. $6\frac{1}{2}$ sq. yds. 6·84 sq. ft. **12.** 28·2744 sq. in.
13. 5857 sq. ft. **14.** 523·16 ft. **15.** 5·657 ft.
16. £141 7*s*. 5·28*d*. **7.** 184·9392 sq. ft. **18.** As 2 : $\sqrt{\pi}$.
19. $1145\frac{5}{11}$. **20.** 45·135 ch. **21.** 12 po.
22. $62\frac{1}{2}$ yds. **23.** 8 ac. 2 ro. 22 po. **24.** 530·73 sq. yds.
25. 4 : 9 : 25. **26.** 126. **27.** £25 4*s*. 8*d*.
28. As 49 : 81. **29.** As 1 : 2. **30.** $5\frac{17}{20}$ yds.
31. $\frac{22}{75}$; $\frac{53}{75}$. **32.** 30·9 sq. ft. **33.** $6\frac{1}{2}$*d*. nearly.
34. 22·42 ft. **35.** £157 4*s*. 2*d*.

EXERCISE XVIII.

1. 100 sq. ft. **2.** 1·254 sq. ft. **3.** 88·802 sq. ft.
4. $157\frac{1}{2}$ sq. ft. **5.** $16\frac{2}{3}$ ft. **6.** $7\frac{7}{16}$ ft. **7.** 15·41 ft.
8. 27·53 sq. in. **9.** $\frac{24}{\sqrt{\pi}}$, or 13·56 ft. nearly.
10. $36\frac{1}{7}$ sq. ft.; 12·04 ft. **11.** 91° 40′ 22·8″.
12. 9·254 sq. ft. **13.** $25\frac{3}{7}$ sq. ft. **14.** 208·2 sq. ft.
15. 13 ft.

EXERCISE XIX.

1. 33·48 sq. ft. **2.** 509.38 sq. ft. **3.** 439·4 sq. ft.
4. 1191·23 sq. in. **5.** $194\frac{1}{4}$ sq. ft. **6.** 100·20 sq. in.
7. 20·382 sq. ft.; 686·478 sq. ft. **8.** 28·05 sq. ft.
9. 524·48 sq. ft. **10.** 2·7 sq. in. **11.** 88·4.
12. 368 sq. in. **13.** $9\frac{3}{7}$ sq. ft. **14.** $\frac{36}{847}$ sq. ch.

EXERCISE XX.

1. 801·1 sq. ft. **2.** 660 sq. ft. **3.** 37 sq. yds. $6\frac{3}{7}$ sq. ft.
4. £4 0*s*. 8*d*. **5.** 24 ft. **6.** 11 sq. in.
7. 432·34 sq. ft. **8.** 235·714 sq. in. **9.** 112° 30′.
10. As 5 : 4. **11.** As 14 : 11. **12.** $7\frac{1}{2}$ ft.

XXV.—MISCELLANEOUS EXERCISES.

I.

1. £162 9*s*. 6*d*. **2.** 859·03 sq. ft. **3.** 5 in.
4. 6 ac. 3 ro. 22 po. $2\frac{1}{2}$ sq. yds. 6 sq. ft. **5.** 210·44 sq. in.
6. 24·3 in.

II.

1. 353 ft. **2.** 18·13 in. **3.** $326\frac{6}{7}$ sq. ft.
4. $1\frac{53}{112}$ sq. ft. **5.** 64 ft. **6.** £1 14*s*. $1\frac{13}{18}$*d*.

III.

1. BD = 1040; DA = 765·5. **2.** 68 sq. yds. 4 sq. ft.
3. $\frac{44}{63}$ sq. ft. **4.** £3 10*s*. **5.** 72 sq. ft. **6.** 2 ro. 5·4 po.

IV.

1. 3 yds. nearly.
2. $194\frac{1}{4}$ sq. ft.
3. 2 ac. 37 po. 6 sq. ft. 108 sq. in.
4. $26\sqrt{3}$ ft.
5. 5·1416 sq. in.
6. £11 16*s.* $7\frac{3}{8}$*d.*

V.

1. 45·4 sq. in. nearly.
2. 96 in.
3. $339\frac{3}{7}$ sq. in.
4. £1210 12*s.* $6\frac{6}{7}$*d.*
5. 820 in.
6. 71·6 sq. ft.

VI.

1. 113·5 sq. in.; 287·5 sq. in.
2. 4 ft. $5\frac{3}{4}$ in.; 13 ft. $5\frac{1}{4}$ in.
3. 296·04 sq. ft.
4. $1\frac{1}{4}$ sq. ft.
5. 5 ft. nearly.
6. 3 sq. ft.

VII.

1. 15 in. nearly.
2. 58·86 ft.
3. 42 ft.
4. 1 ac. 3 ro. 32 po. 2 sq. yds. 6 sq. ft.
5. ·81 sq. ft.
6. 61·237 ft

VIII.

1. 15 in.
2. 0·03 sq. in.
3. As 57 : 50.
4. £43 3*s.* 4*d.*
5. 11·78 yds.
6. £5 15*s.*

IX.

1. 7956 sq. ft; 51 ft.
2. 54·9376 sq. ft.
3. As 14 : 11.
4. £4 15*s.* $8\frac{1}{7}$*d.*
5. 1·18 sq. ft.
6. $32\frac{1}{2}$ in.

X.

1. 97·2.
2. 14 in.
3. 15394.
4. 3 ro. 1·78 po.
5. 2·16 ac.; 3·84 ac.
6. $10\frac{1}{2}$ in.; $8\frac{1}{2}$ in.

EXERCISE XXI.

1. 18 cub. ft. 1664 cub. in.
2. 658 cub. ft. 936 cub. in.
3. 1049760 cub. in.
4. 75 cub. yds. 800 cub. in.
5. 4159·11 cub. in.
6. 1 ton 8 cwt. 2 qrs. 8 lbs.
7. 62109·776 cub. in.
8. 578·5 ozs.
9. 3·9 cub. in.
10. 200 cub. ft. 200 cub. in.
11. 32 cub. ft. 752 cub. in.
12. $29\frac{1}{16}$ sq. ft.
13. 42 sq. ft.
14. 52 cub. in.
15. $265\frac{5}{8}$ lbs.
16. 224 gals.
17. $10\frac{29}{108}$ lbs.
18. 34·6592 cub. in.
19. 6 cwt. 78 lbs.
20. $9\frac{1}{3}$ gals.
21. ·438375 cub. in.
22. 160 cub. ft. 770 cub. in.
23. 64 cubes.
24. 4 ft. $3\frac{3}{4}$ in.

EXERCISE XXII.

1. 78 cub. ft. 810 cub. in.
2. $3605\frac{1}{27}$ cub. ft.
3. 1124 cub. ft. 228 cub. in.
4. 20 cub. ft. 1440 cub. in.
5. 4 cwt. 3 qrs. $4\frac{1}{4}$ lbs.
6. 4·79 in.
7. 89 cwt.
8. 934 gals.

9. 6 tons nearly. **10.** 4 cwt. 1 qr. 24 lbs. **11.** 83 ft. 5 in.
12. 13 tons 17 cwt. 3 qr. nearly. **13.** $\frac{5}{256}$ in.
14. $\frac{7}{10}\sqrt{10}$ lbs. **15.** $12\sqrt{3}$ ft. **16.** $5\sqrt{3}$ ft. **17.** $124\frac{17}{72}$ sq. ft.
18. $204\frac{1}{6}$ sq. ft. **19.** 2 ft. **20.** 60 ft. **21.** 27 in.
22. 189 in. **23.** 11·514 cub. ft. **24.** £4 5*s.* $6\frac{3}{4}$*d.* **25.** $16\frac{4}{7}$ ft.
26. 284·92 gals. **27.** $\frac{125}{9}\sqrt{3}$ ft. **28.** 17·32 ft. **29.** $10\sqrt{3}$ ft.
30. 261·7 gals.

EXERCISE XXIII.

1. 480 cub. ft. **2.** 100 cub. ft. **3.** $1\frac{1}{4}$ cub. ft. **4.** $\frac{5}{9}$ cub. ft.
5. $\frac{5}{18}$ cub. ft. **6.** $7\frac{1}{2}$ cub. ft. **7.** 9 cub. ft. **8.** $2\frac{5}{8}$ sq. ft.
9. 6 ft. **10.** $\frac{5}{36}$ cub. ft. **11.** $\frac{25}{288}$ cub. ft.

EXERCISE XXIV.

1. $9\frac{1}{2}$ sq. ft. nearly. **2.** $67\frac{1}{2}$ sq. ft. **3.** 847 sq. ft.
4. $172\frac{6}{7}$ sq. ft. **5.** $9\frac{3}{7}$ cub. ft. **6.** $247\frac{1}{2}$ cub. ft.
7. $4\frac{12}{77}$ ft. **8.** $2614\frac{6}{7}$ sq. ft. **9.** 14·235 cub. ft.
10. £6 13*s.* $5\frac{1}{2}$*d.* **11.** 177·014 sq. ft. **12.** $\frac{5}{6}$ cub. ft.
13. $51\frac{9}{16}$ cub. ft. **14.** 251·328 sq. ft.
15. 5 cub. yds. 9 cub. ft. 648 cub. in. **16.** £26 14*s.* **17.** 408 sq. in.
18. 20 pts. **19.** 8·94 sq. ft.

EXERCISE XXV.

1. 109 cub. ft. 133 cub. in. **2.** 1 cub. ft. 162 cub. in.
3. 83·449 cub. ft. **4.** 6·46 ft. **5.** $5\frac{1}{4}$ cub. ft. **6.** $55\frac{11}{16}$ cub. ft.
7. $17\frac{1}{2}$ sq. ft. **8.** 83·136 cub. ft. **9.** 180 cub. in.
10. $18\frac{6}{7}$ cub. ft. **11.** 1485 sq. in. **12.** $7\frac{6}{7}$ cub. ft.
13. 8 sq. ft. **14.** 24 cub. ft. **15.** $3\frac{1}{5}$ sq. ft. **16.** 2·078 cub. ft.

EXERCISE XXVI.

1. 217·3 cub. in. **2.** $\frac{242}{875}$ cub. in. **3.** $\frac{55}{56}$ cub. ft.
4. ·0103 cub. in. **5.** $\frac{55}{112}$ cub. ft. **6.** $71\frac{2}{3}$ cub. in. nearly.
7. $79\frac{1}{5}$ cub. in. **8.** 951·6 cub. in. **9.** $1211\frac{1}{7}$ cub. in.
10. 491 cub. in. nearly. **11.** $2\frac{37}{42}$ cub. ft. **12.** 1063 cub. in.
13. 6·54 cub. ft. **14.** 174 cub. in. nearly.
15. Flat ring is $\frac{287}{392}$ cub. ft. larger.

EXERCISE XXVII.

1. $49\frac{7}{12}$ sq. ft. **2.** 974·25 cub. ft. **3.** 5542·4 cub. ft.
4. 4156·8 cub. ft. **5.** 3466145 cub. yds. **6.** £16 13*s.* 4*d.*
7. As 1 : 7. **8.** 130·98 cub. ft. **9.** 744 sq. ft.
10. 311·769 cub. ft. **11.** 737·2 cub. ft. **12.** 60 sq. ft.
13. 110·52 sq. in. **14.** $26\frac{2}{3}$ cub. ft. **15.** 98 sq. ft.
16. 400 cub. ft. **17.** 63·26 sq. yds. **18.** 2500 cub. ft.
19. Equal **20.** As $a : b$. **21.** As 19 : 8.
22. As 91 : 117 : 8. **23.** $181\frac{1}{3}$ cub. ft. **24.** £28 2*s.* 4*d.*

EXERCISE XXVIII.

1. 611·9 sq. ft. **2.** 83·8 yds. **3.** $600\frac{3}{49}$ cub. ft.
4. $183\frac{1}{3}$ cub. ft. **5.** 420·8 sq. ft. **6.** 70·7 cub. ft.
7. 12 ft. 4 in. **8.** £19 11*s.* $1\frac{1}{4}$*d.* **9.** 265·14 cub. ft.
10. 202·65 cub. ft. **11.** 19·24 cub. ft. **12.** £30.
13. £7 14*s.* 6*d.* **14.** 56·83 cub. ft. **15.** £4 12*s.* $2\frac{1}{2}$*d.*
16. 10·77 in. **17.** 2 sq. ft. 45·8 sq. in. **18.** $74\frac{23}{24}$ cub. ft.
19. $8\frac{8}{21}$ cub. ft. **20.** As 7 : 1. **21.** As 19 : 7 : 1.
22. $89\frac{5}{6}$ cub. ft.

EXERCISE XXIX.

1. $1\frac{41}{72}$ sq. ft. **2.** $1844\frac{1}{3}$ cub. yds. **3.** 30402·8 gals.
4. 760 cub. in. **5.** 76·76 cub. ft. **6.** 37000 cub. ft.
7. $41\frac{1}{4}$ sq. ft. **8.** $57\sqrt{3}$ cub. ft. **9.** 138 sq. ft.
10. 296050·5 gals. **11.** £419 3*s.* 10*d.* **12.** 176 sq. ft.

EXERCISE XXX.

1. 1611·5 sq. ft. **2.** 1186·6 cub. ft. **3.** $35\frac{5}{8}$ tons nearly.
4. 4015 cub. ft. **5.** 35·6 sq. ft. **6.** 7 cub. ft. nearly.
7. 74·4 cub. ft. **8.** 200 sq. ft. nearly. **9.** 1779·9 cub. ft.
10. $447\frac{1}{3}$ cub. ft. **11.** 154·7 gals. **12.** 2·912 tons.
13. 85 sq. in. **14.** 20·36 sq. ft. **15.** 16·54 cub. ft.
16. 23·88 cub. ft. **17.** 251·8 gals.

EXERCISE XXXI.

1. 45 cub. in. **2.** 320 cub. in. **3.** 84 cub. in.
4. 112 cub. in. **5.** 92 sq. in. **6.** 5·56 in.
7. $29\frac{1}{3}$ cub. in. **8.** $405\frac{1}{3}$ cub. in. **9.** 1172·8 cub. yds.
10. $16\frac{5}{9}$ cub. yds. **11.** 24954 cub. yds. **12.** £3989 17*s.* 6*d.*
13. 500 cub. in.

EXERCISE XXXII.

1. 1·26 ft. **2.** 1·09 ft. **3.** 8·32 ft. **4.** 18·9.
5. 4·018 in. **6.** 11·11 ft. **7.** As 1 : 7. **8.** $4\frac{4}{9}$ in.
9. 16·207 in. **10.** As 64 : 125. **11.** 6 in.
12. As 2 : 3. **13.** $10\sqrt[3]{3}$ in. **14.** 6*s.* 8*d.*
15. $V : V_1 :: 4 : 5$ inversely as heights.
16. $S : S_1 :: 4 : 3$ inversely as radii.
17. 17·4 in. **18.** 51 cub. ft. **19.** As 125 : 64.
20. Height of cylinder : height of prism :: $21\sqrt{3}$: 275.
21. As 1 : 7.

EXERCISE XXXIII.

1. 154 sq. ft. **2.** 3·09 ft. **3.** $65\frac{10}{21}$ cub. ft.

4. 10·7 in.
5. 94 cub. in.
6. 1353 cub. in.
7. 13·64 sq. ft.
8. 22·3 in.
9. $709\frac{1}{2}$ lbs.
10. 823·2192 cub. in.
11. 1 in.
12. $\frac{1}{4}$ of the surface.
13. £237 10*s*.
14. $\frac{3}{800}$.
15. $2932\frac{1}{3}$.
16. $\frac{2}{\pi\sqrt[3]{20}}$ ft. = 2·8 in.
17. 1657·4 ft.
18. As 16 : 15.
19. 12 lbs.
20. 15·708 ft.
21. As $\sqrt{21} : \sqrt{11}$.
22. 9 in.
23. C : S : : $\sqrt[3]{21} : \sqrt[3]{11}$.
24. 25 sq. in.
25. 15 ft.
26. $3\sqrt{3} : 1$.
27. 2·44 in.
28. 1, $(\frac{3}{11})^2$, $(112)^2$.

EXERCISE XXXIV.

1. 12π sq. in.
2. 68π sq. ft.
3. $\frac{25}{18\pi}$ in.
4. $\frac{3}{112}\pi r^2$ tons.
5. 48π sq. ft.
6. 5236 cub. ft.
7. 208·59 cub. ft.
8. 2 ft.
9. $\frac{8000}{3}$ miles.
10. 51·07 sq. ft.
11. 308·43 sq. ft.
12. $\frac{4000}{7}$ miles.
13. 126·2 mls.
14. 22·07 sq. ft.
15. $10\sqrt{2}$.
16. $\sqrt[3]{10}$.
17. 8 ft.
18. $\frac{1}{16}$.
19. 32·4 cub. ft.
20. $62\frac{17}{84}$ cub. ft.
21. $11\frac{11}{56}$ cub. ft.

EXERCISE XXXV.

1. £5 5*s*. $4\frac{1}{4}$*d*.
2. 4*s*. $6\frac{1}{2}$*d*.
3. £2 12*s*. $10\frac{1}{4}$*d*.
4. £8 3*s*. 10*d*.
5. £12 12*s*.
6. £326 8*s*.
7. $75\frac{5}{8}$ cub. ft.
8. £161 8*s*.
9. £233 17*s*. 9*d*.; 864 planks.
10. £3 15*s*. $7\frac{1}{2}$*d*.
11. £15 15*s*.
12. $128\frac{1}{4}$ cub. ft.
13. 11·22 ro.
14. £144.
15. £106 17*s*. 6*d*.
16. £30 15*s*.
17. 50 cub. ft.
18. £46 8*s*. $1\frac{1}{2}$*d*.
19. $27\frac{7}{9}$.
20. $33\frac{3}{4}$ cub. ft.
21. 48 cub. ft.
22. $11\frac{167}{363}$ ro.
23. £55.
24. £36.
25. £5 11*s*. $5\frac{1}{7}$*d*.

XLII.—MISCELLANEOUS EXERCISES ON SOLIDS.

I.

1. 6·45 ft.
2. 19·428 in.
3. $3788\frac{8}{9}$ cub. yds.
4. 93·04 lbs.
5. 17*s*. $3\frac{7}{9}$*d*.
6. $477\frac{3}{4}$ cub. in.

II.

1. 9·34 ft.
2. 71·574 cub. ft.
3. 17 ft.; $25\frac{1}{2}$ ft.
4. 17 cwt. 21 lbs.
5. 385·6 gals.
6. $4826\frac{1}{4}$ cub. ft.

III.

1. $1\frac{13}{32}$ cub. ft.
2. £2 15*s*.
3. 9 sq. ft.
4. H = D.
5. 2017·8 gals.
6. £29 14*s*.

IV.

1. 12 ft. 3 in. 2. 7·81 ft. 3. 11·46 sq. in.
4. 249·4152 cub. ft. 5. 5·0316 miles 6. 6 sq. ft.; 4·836 sq. ft.

V.

1. 103·92 ft. 2. 2047 cub. ft. 1572 cub. in.
3. 43552 gals. 4. π cub. ft. 5. One-half the radius.
6. Cylinder = 18·85 sq. ft.; sphere = 16·46 sq. ft.

VI.

1. £5 9*s.* $1\frac{1}{2}$*d.* 2. 7·45 in. 3. 8·01 ft.
4. $601\frac{1}{2}$ lbs. 5. 1977·9 cub. ft.
6. Cube, 1 cub. ft.; sphere, 1·382 cub. ft.

VII.

1. 50·2656 lbs. 2. 29·96. 3. 83 : 518 : 399.
4. 103·4 gals. 5. As 3 : 5. 6. 796 in.

VIII.

1. $2\frac{1427}{3168}$ in. 2. $10\frac{1}{3}$ sq. ft. 3. 7 ft. $2\frac{4}{5}$ in.
4. 50 ft. 5. 99·471 in. 6. As 1 : 1·2407.

IX.

1. $34\frac{142}{377}$ in. 2. 15 ft. 9 in.; 16 ft. 2 in. Yes.
3. 9 cwt. 1 qr. 26 lbs. $0\frac{3}{16}$ oz. 4. 26·272 cub. in.
5. $5\frac{1}{2}$ in. 6. 1 ft. 11 in. deep, 5 ft. 9 in long.

X.

1. 127·234 cub. ft. 2. 83·026 cub. ft. 3. 157·08 sq. ft.
4. 1·1412 cub. ft. 5. £103 12*s.* $7\frac{751}{834}$*d.* 6. 584 lbs.

XLIII.—MISCELLANEOUS EXERCISES (GENERAL).

I.

1. $226\frac{2}{7}$ sq. in. 2. 100 tons nearly. 3. 39172 cub. yds.
4. 18 miles. 5. £42 15*s.* 6. $662\frac{53}{56}$ gals.

II.

1. 6 ft. 2. 18327 ft. 3. 1 : 506880.
4. $44196\frac{3}{7}$ tons 5. 5028·56 cub. ft. 6. 78 yds. and 58 yds.

III.

1. 46·765 in. 2. £24 10*s.* 3. 137 sq. ft. 127·69 sq. in.
4. $4\frac{1}{2}$ ft. 5. 864. 6. 5·1416 sq. in.

IV.

1. $1\dot{3}$ and $1\cdot\dot{7}$.	2. 25 ch.	3. 5 ft.
4. 904·78 sq. in.	5. $\frac{1}{16}$ sq. in.	6. 509·208 cub. in.

V.

1. 595 ft.	2. 8·5942 cub. ft.	3. 6·349 ft.
4. £5 18*s.* $11\frac{1}{7}$*d.*	5. 3.	6. £2 5*s.* $1\frac{7}{8}$*d.*

VI.

1. $9 : 4\sqrt{3}$.	2. 3, 5, 7.	3. 1728.
4. 323·84 yds	5. 472 sq. ft.	6. £14 17*s.*

VII.

1. 3 ac. 3 ro. 35 po.	2. 151·2 in.	3. 2673 cub. in.
4. 486 sq. yds.	5. 6 ft.; $4\frac{1}{2}$ ft.; 3 ft	6. 4318934·235 sq. ft.

VIII.

1. 2·52.	2. 9360 cub. ft.	3. 15·15 lbs.
4. 1*s.* 3*d.*	5. 171·709 sq. ft.	6. 12·5664 sq. ft.

IX.

1. ·577 and ·816 ft.	2. 6 yds.; 1 yd.	3. $754\cdot7\dot{2}$ sq. ft.
4. 66 lbs. $10\frac{181}{324}$ ozs.	5. £6 17*s.* $9\frac{3}{4}$*d.*	6. 3 ft. 0·48 in.

X.

1. 3·39 in.	2. 2513·28 cub. ft.	3. 243·55 sq. ft.
4. 2·315 hrs.	5. $1607\frac{1}{7}$ cub. in.	6. 168 sq. in.

XLIV.—EASY EXERCISES.

I.—On Measurements.

A.

1. 20 sq. ft.	2. 24 cub. ft.	3. 110 yds.
4. 360 ch.	5. 484 sq. yds.	6. 10 sq. ch.

B.

1. $4\frac{1}{2}$ ac.	2. $759\frac{1}{2}$ ac.	3. $3\frac{3}{4}$ ac.
4. 198 ft.	5. 1870 yds.	6. 10 ch.

C.

1. 60 sq. ft.	2. $612\frac{1}{2}$ sq. ft.	3. 825 sq. yds.
4. 53 ft.	5. 77 sq. ft.	6. 75 sq. ft.

II.—On Parallelograms.

A.

1. Length × perpendicular height = area. 2. 20 sq. yds.
3. 1050 sq. in. 4. $131\frac{1}{4}$ ac. 5. 30 yds. 6. 1125 sq. yds.

B.

1. Square the side. 2. Area = one-half of the square of the diagonal.
3. $2\frac{1}{2}$ ac. 4. 72 sq. ft. 5. £13 6*s*. 8*d*. 6. 352 yds.

C.

1. Take the square root of the area.
2. Take the square root of twice the area. 3. 22 yds.
4. 110 yds. 5. 220 yds. 6. £4 7*s*. 6*d*.

D.

1. Divide the area by the side 2. 36 ft.
3. 4 ft. 4. 18 ft. 5. £4 16*s*. 6. 7*s*. 6*d*.

III.—On Triangles.

A.

1. Take the square root of the sum of the squares of the base and perpendicular. 2. 15 ft. 3. 20 ft.
4. 50 miles. 5. 880 yds. 6. $112\frac{1}{2}$ sq. ft.

B.

1. Take one-half of the product of the base and perpendicular.
2. $\frac{1}{4}$ ac. 3. 75 ft. 4. 25*s*. 5. 119. 6. 5*s*. 4*d*.

C.

1. Let $s = \dfrac{a+b+c}{2}$ Then area $= \sqrt{s \cdot \overline{s-a} \cdot \overline{s-b} \cdot \overline{s-c}}$

2. 15·2 ft. 3. 11·6 sq. in. 4. 15 ft. 5. 15 ft. 6. $2\frac{2}{5}$ ft.

IV.—Miscellaneous.

A.

1. $7\frac{1}{4}$ ac. 2. 119 sq. ft. 3. 3586 yds.
4. $46\frac{4}{5}$ sq. yds. 5. 20 yds. 6. 8 ft.

B.

1. 294 sq. yds. 2. 26 sq. ft. 3. 170 sq. in.
4. 352 yds. 5. 15 ft. 6. 36 ft.

V.—On Circles.

A.

1. Multiply the diameter by π.
2. The ratio of circumference to diameter = $3\frac{1}{7}$.
3. 22 ft. 4. 22 ft. 5. 11 ft. 6. 110 yds.

B.

1. Multiply the square of the diameter by one-fourth of π.
2. Multiply the square of the radius by π.
3. $9\frac{5}{8}$ sq. ft. 4. $38\frac{1}{2}$ sq. ft. 5. $12\frac{4}{7}$ sq. ft. 6. 88 ft.

C.

1. Divide the circumference by π.
2. Divide the circumference by 2π.
3. 3 ft. 4. $10\frac{1}{2}$ ft. 5. $14\frac{2}{3}$ ft. 6. 560 yds.

D.

1. Divide the square of the circumference by 4π. 2. $50\frac{10}{11}$ ac.
3. $38\frac{1}{2}$ sq. yds. 4. As 11 : 14. 5. As 11 : 14. 6. $78\frac{4}{7}$ sq. ft.

E.

1. Take twice the square root of the area divided by π.
2. Take the square root of the area divided by π.
3. 14 yds. 4. 2 ft. 5. 6 ft. 6. As 7 : 22.

F.

1. Twice the square root of π times the area. 2. 44 yds.
3. 22 yds. 4. 11 ft. 5. As $2\sqrt{2} : 3$. 6. 220 yds.

G.

1. Multiply the product of the sum and difference of the outer and inner radii by π.
2. 330 sq. in. 3. 462 sq. ft. 4. $7\frac{6}{7}$ ac. 5. 10 ft. 6. 960.

VI.—On Sectors.

A.

1. A surface bounded by two radii of a circle and the arc of the circle between these radii.
2. Take one-half of the product of the length of the arc and the radius.
3. 35 sq. ft. 4. $17\frac{1}{2}$ sq. yds. 5. 100 sq. ft. 6. $38\frac{1}{2}$ sq. ft.

B.

1. $27\frac{1}{2}$ sq. ft. 2. 44 sq. ft. 3. 90°.
4. 120°. 5. 120 sq. ft. 6. $173\frac{1}{4}$ sq. ft.

VII.—On Chords and Segments.

1. A line drawn through a circle terminated each way by the circumference.
2. A surface bounded by a chord and the arc which the chord cuts off.
3. 8 ft. 4. 5 ft. 5. 5 ft. 6. 4 ft.

VIII.—On the Ellipse.

1. Multiply the product of the two diameters by one-fourth of π.
2. 55 sq. ft. 3. 132 sq. ft. 4. 11 ac.
5. 264 sq. ft. 6. As 7 : 6.

IX.—Miscellaneous.

1. 12$\frac{4}{5}$ ac.	2. 2 ft. 1 in.	3. £6.
4. 10$\frac{1}{2}$ ft.	5. 12 ft.	6. 5 ac.

X.—On Standards of Capacity.

1. 10 lbs.	2. 277$\frac{1}{4}$ cub. in.	3. 62$\frac{1}{2}$ lbs.
4. 6$\frac{1}{4}$ gals.	5. 1$\frac{1}{4}$ lbs.	6. 34$\frac{21}{32}$ cub. in.

XI.—On the Parallelopiped.

A.

1. 64 cub. ft.	2. 96 sq. ft.	3. 500 lbs.
4. 9 cub. ft.	5. 27 sq. ft.	6. 6 yds.

B.

1. 6 yds.	2. 2400 cub. ft.	3. 10 in. cube.
4. 1$\frac{1}{3}$ ft.	5. 1155$\frac{9}{11}$ gals.	6. 750 lbs.

C.

1. 72.	2. 15*s.*	3. 562$\frac{1}{2}$ tons.
4. 108.	5. 37 sq. ft.	6. 2 ft.

XII.—On Prisms and Cylinders.

1. Multiply the area of the base by the perpendicular height.
2. 144 cub. in.
3. 96$\frac{1}{4}$ cub. ft.
4. £4 10*s.* 9*d.*
5. 1200 gals.
6. 12$\frac{1}{2}$ cub. ft.

XIII.—On Cones and Pyramids.

1. Multiply the area of the base by one-third of the perpendicular height.
2. As 1 : 3.
3. Take the square root of the sum of the perpendicular and radius of the base.
4. Multiply the circumference of the base by half the slant height.
5. 600 cub. in.
6. 48 cub. ft.

XIV.—Miscellaneous.

1. 25 ft.	2. 675 sq. ft.	3. 440 sq. ft.
4. 385 cub. ft.	5. 30 cub. in.	6. 21$\frac{31}{64}$ tons.

XV.—On the Sphere.

1. Multiply the cube of the diameter by one-sixth of π.
2. As 21 : 11.
3. As 2 : 3.
4. 22$\frac{11}{24}$ cub. ft.
5. 4$\frac{4}{21}$ cub. ft.
6. As 21 : 11.

XVI.—Miscellaneous.

A

1. As 9 : 16. 2. As 27 : 64. 3. As 9 : 16.
4. As 27 : 64. 5. As $7\sqrt{3}$: 33. 6. As 8 : 27.

B.

1. $25\sqrt{2}$. 2. $\frac{50\sqrt{3}}{3}$ and $\frac{50}{3}\sqrt{6}$.
3. $5\frac{1}{4}$ sq. ft. 4. $28\frac{2}{7}$ sq. ft. 5. $188\frac{4}{7}$ sq. ft. 6. 200 miles.

C.

1. Take one-balf the product of the two diagonals. 2. 35 sq. ft.
3. 10·825 sq. ft. 4. 2·598 sq. ft. 5. As 16 : 25.
6. $30\frac{3}{8}$ sq. ft.

D.

1. As 1 : 36. 2. As 1 : 660. 3. 10, 15.
4. As 1 : 2504. 5. 3 : $2\sqrt{3}$. 6. 30.

E.

1. 2·5 ft. 2. 12·5 sq. ft. 3. 520 sq. ft.
4. 2 : $\sqrt[4]{27}$. 5. 20 sq. yds. 6. $19\frac{9}{14}$ sq. ft.

F.

1. 1540 cub. ft. 2. 14 yds. 3. 88 sq. ft.
4. 110 sq. ft. 5. 616 sq. ft. 6. 132 sq. ft.

G.

1. As $22^3 : 7^3$. 2. £3. 3. 15 ft.
4. As 21 : 11. 5. 30 cub. ft. 6. 56 miles.

H.

1. 80 sq. yds. 2. 141·4 yds. 3. 462 sq. ft.
4. 34·65 ac. 5. 60 plots. 6. 480 times.

I.

1. $23\frac{1}{3}$ sq. yds. 2. 58 ft. 3. $64\frac{4}{9}$ sq. yds.
4. £4 1*s*. 8*d*. 5. £1 9*s*. $3\frac{1}{9}$*d*. 6. £7.

K.

1. 154 sq. ft. 2. 2·8 cub. ft. 3. 132 sq. ft.
4. £11 11*s*. 5. $21\frac{3}{14}$ cub. ft. 6. 14 ft.

XLVIII.—HARD EXERCISES.

1. 9 : 4. **2.** 8 in. **3.** 30350·9.
4. 27 cub. ft **5.** 106·7 cub. yds. **6.** 491·07.
7. 19·59 in. **8.** 7·3416 ft. **9.** 600 cub. ft.
10. 1·964 in. **12.** 196·4 yds. **13.** 23·91 ac.
14. $\frac{1}{2}$ cub. ft. **15.** 2544·69 cub. ft. **16.** The former.
17. 82·729 cub. in.; 169·64 sq. in. **18.** 282·744 cub. in.
19. 78·617 sq. in. **20.** 180 yds. 5 in. **21.** 0·0404; 0·0805.
22. 300. **23.** 9 in.; 438·9 cub. in. **24.** 0·9427 sq. in.
25. 92·1533 cub. in. **26.** 10686 sq. yds. **27.** 0·463 in.
28. 460047 tons. **29.** 30·041 in. **30.** £127 4*s.* 10·2*d.*
31. 12612600 gals.; 29594565000 tons. **32.** 2·545 ft. per sec.
33. 49·582 cub. in. **34.** 119·43 cub. in.; 147·714 sq. in.
35. 77·8 miles. **36.** 25·457 cub. in. **37.** 1·0272 sq. in.
38. 3·464 sq. in.; 121·736 cub. in. **39.** 565·488 sq. ft.
40. 7·444 in. **41.** 740·6 miles an hour. **42.** 0·9317 in.; 2·5 in.
43. 5·823 in. **44.** 326·858 sq. in. **45.** 125·5 cub. in.
46. 4·626 in. **47.** $\frac{1}{3}$; 70° 33′. **48.** 766·44 yds.
49. 25$\frac{2}{3}$ cub. in. **50.** 3854·6 cub. ft.; 3·94 ft.
51. 307·88 sq. in.; 494·28 cub. in. **52.** 10600 cub. ft.
53. 9·97 in. **54.** 95·23 sq. miles. **55.** 11·7 miles.
56. 896·76 cub. in. **57.** 163·43 sq. in.; 79·62 cub. in.
58. 21·5 per cent. **59.** 20828300. **60.** 1·63 in.
61. 2·28 cub. in; 61·56 cub. in. **62.** 18·92 miles. **63.** 6·4638
64. 3·92 in. **65.** 436·5 miles. **66.** 2·0926 ft.

SANDHURST.

1. 1949·7 sq. in.; 1949·7 cub. in. **2.** 3417·825 sq. ft.
3. 950 sq. yds. **4.** 9·4248 cub. ft. **5.** 183$\frac{1}{3}$ sq. in.
6. 0·5418 ft. **7.** 1612·5 sq. ft. **8.** 4·1888 cub. in.; 5·236 cub. in.
9. £833 17*s.* 3*d.* **10.** 143 yds. **11.** 10$\frac{1}{2}$ in.
12. 93·976 sq. in. **13.** 1889·7 sq. ft.
14. 92·4 sq. in.; 14·941 in. **15.** 20 ft. **16.** 1·3 ft.; 8·168 sq. ft.
17. 4 in. **18.** 1·245 in. **19.** 1 in.
20. 933·55 cub. in.; 1381·70 cub. in.

WOOLWICH.

1. 122·45 ft. 2. 45°. 3. 233·5 ft.; 239·05 tons.
4. 165·748. 5. 0·563119 ft. 6. £19958.
7. 1221·7 cub. in. 8. 190·76 sq. ft. 9. 7·43 ft.
10. 96 cub. in.; 138·53 sq. in. 11. 2·338 tons. 12. 209·6 sq. yds.
13. 13·27 sq. ft. 14. 22·66 sq. yds. 15. 2·737 in.